Maxima 在微積分上之應用

無限的數列

國立屏東教育大學 應用數學系 研究助理 徐偉玲

weilinghsu@mail.npue.edu.tw

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除另有說明外，本文件採用創用 CC「姓名標示、非商業性」

2.5 台灣條款

9.1 Sequences

Example 1.

If the sequence is simple enough one can look at the first few terms and guess the general rule for computing the n th term. For instance：

1, 1, 1, 1, 1,…	$a_n = 1$
-1, 0, 1, 2, 3, …	$a_n = n-2$
-2, -4, -6, -8, -10,…	$a_n = -2n$
1, -1, 1, -1, 1,…	$a_n = (-1)^{n-1}$
1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$,…	$a_n = \frac{1}{n}$

Solution：

1, 1, 1, 1, 1,…

(%i1) load(solve_rec); 因 Maxima 並不會自動做解遞迴式，於是我們必須讀取模組，所謂模組就是 Maxima 內建的一小段程式，在此我們要讀取 solve_rec 這個模組，來做遞迴式的解 //讀取模組 solve_rec

```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/solve_rec/
```

(%i2) solve_rec(a[n+1]=a[n],a[n],a[1]=1); 解遞迴式指令：solve_rec(遞迴式，變數，初始值) //遞迴式由題目推知 $a_{n+1} = a_n$ ，變數所求為 a_n ，初始值為 1

(%o2) $a_n = 1$

-1, 0, 1, 2, 3, …

(%i1) load(solve_rec); 因 Maxima 並不會自動做解遞迴式，於是我們必須讀取模組，所謂模組就是 Maxima 內建的一小段程式，在此我們要讀取 solve_rec 這個

模組，來做遞迴式的解　　//讀取模組 solve_rec

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/solve_rec/

(%i2) solve_rec(a[n+1]=a[n]+1,a[n],a[1]=-1); 解遞迴式指令：solve_rec(遞迴式，變數，初始值)　//遞迴式由題目推知 $a_{n+1}=a_n+1$，變數所求為 a_n，初始值為-1

(%o2) $a_n = n - 2$

-2, -4, -6, -8, -10,…

(%i1) load(solve_rec); 因 Maxima 並不會自動做解遞迴式，於是我們必須讀取模組，所謂模組就是 Maxima 內建的一小段程式，在此我們要讀取 solve_rec 這個模組，來做遞迴式的解　　//讀取模組 solve_rec

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/solve_rec/

(%i2) solve_rec(a[n+1]=a[n]-2,a[n],a[1]=-2); 解遞迴式指令：solve_rec(遞迴式，變數，初始值)　//遞迴式由題目推知 $a_{n+1}=a_n-2$，變數所求為 a_n，初始值為-2

(%o2) $a_n = -2(n-1) - 2$

(%i3) ratsimp(%);　//將上式做整理得到 $a_n = -2n$

(%o3) $a_n = -2\ n$

1, -1, 1, -1, 1,…

(%i1) load(solve_rec); 因 Maxima 並不會自動做解遞迴式，於是我們必須讀取模組，所謂模組就是 Maxima 內建的一小段程式，在此我們要讀取 solve_rec 這個

模組，來做遞迴式的解　　//讀取模組 solve_rec

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/solve_rec/

(%i2) solve_rec(a[n+1]=a[n]*(-1),a[n],a[1]=1);　解遞迴式指令：solve_rec(遞迴式，變數，初始值)　//遞迴式由題目推知 $a_{n+1} = a_n * (-1)$，變數所求為 a_n，初始值為 1

(%o2) $a_n = (-1)^{n-1}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

(%i1) load(solve_rec);　因 Maxima 並不會自動做解遞迴式，於是我們必須讀取模組，所謂模組就是 Maxima 內建的一小段程式，在此我們要讀取 solve_rec 這個模組，來做遞迴式的解　　//讀取模組 solve_rec

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/solve_rec/

(%i2) solve_rec(a[n+1]=a[n]/(1+a[n]),a[n],a[1]=1);　解遞迴式指令：solve_rec(遞迴式，變數，初始值)　//遞迴式由題目推知 $a_{n+1} = \frac{a_n}{1+a_n}$，變數所求為 a_n，初始值為 1

(%o2) $a_n = \frac{n+1}{n} - 1$

(%i3) ratsimp(%);　//將上式做整理得到 $a_n = \frac{1}{n}$

(%o3) $a_n = \frac{1}{n}$

Example 2.

The sequence

3.1, 3.14, 3.141, 3.1415, 3.14159,…

is defined by the rule

$a_n = \pi$ to n decimal places,

that is, $a_n = \frac{m}{10^n}$ where m is the integer such that $\frac{m}{10^n} \le \pi \le \frac{m+1}{10^n}$.

Example 3.

The number $n\,!$, read n factorial, is defined as the product of the first n positive integers;

$n\,! = 1 \cdot 2 \cdot ... \cdot n$

$< n\,! >$ is an important sequence. Its first few terms are

1, 2, 6, 24, 120, 720,…

By convention, $0\,!$ Is defined by $0\,! = 1$.

Example 1. (Continued)

$\lim_{n\to\infty} 1 = 1$, converges, because $a_H = 1$ for all H.

$\lim_{n\to\infty} n - 2 = \infty$, diverges, because $H - 2$ is positive infinite for all H.

$\lim_{n\to\infty}(-2n) = -\infty$, diverges, because $-2H$ is negative infinite for all H.

$\lim_{n\to\infty}(-1)^n$ is undefined, diverges, because $(-1)^{2H} = 1$ but $(-1)^{2H+1} = -1$.

$\lim_{n\to\infty}\frac{1}{n} = 0$, converges, because $\frac{1}{H}$ has standard part zero.

Solution：(%i1) limit(1,n,inf);   極限指令：limit(方程式，極限變數，範圍)   //此例定義之方程式爲1，極限變數爲 n，範圍爲 n 趨近於∞

(%o1) 1

(%i2) limit(n-2,n,inf);   極限指令：limit(方程式，極限變數，範圍)   //此例定義之方程式爲 n-2，極限變數爲 n，範圍爲 n 趨近於∞

(%o2) ∞

(%i3) limit((-2)*n,n,inf);   極限指令：limit(方程式，極限變數，範圍)   //此例定義之方程式爲-2n，極限變數爲 n，範圍爲 n 趨近於∞

(%o3) $-\infty$

(%i4) limit((-1)^n,n,inf);   極限指令：limit(方程式，極限變數，範圍)   //此例定義之方程式爲 $(-1)^n$ ，極限變數爲 n，範圍爲 n 趨近於∞

(%o4) *ind*

(%i5) limit(1/n,n,inf);   極限指令：limit(方程式，極限變數，範圍)   //此例定義之方程式爲 $\frac{1}{n}$ ，極限變數爲 n，範圍爲 n 趨近於∞

(%o5) 0

Example 2. (Continued)

The sequence

3.1, 3.14, 3.141, 3.1415, 3.14159, …, $a_n, \ldots$

where $a_n = (\pi$ to n decimal places), converges to π. That is

$$\lim_{n \to \infty} a_n = \pi .$$

Example 3. (Continued)

$\lim_{n\to\infty} n! = \infty$.

Solution：(%i1) limit(n!,n,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例定義之方程式為 n ！，極限變數為 n，範圍為 n 趨近於∞

(%o1) ∞

Example 4.

$\lim_{n\to\infty} \frac{4n^2+1}{n^2+3n} = \lim_{n\to\infty} \frac{4x^2+1}{x^2+3x} = 4$.

Similarly, if $\lim_{x\to\infty} f(x) = \infty$ then $\lim_{n\to\infty} f(n) = \infty$.

Solution：(%i1) limit((4*n^2+1)/(n^2+3*n),n,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例定義之方程式為 $\frac{4n^2+1}{n^2+3n}$，極限變數為 n，範圍為 n 趨近於∞

(%o1) 4

Example 5.

$\lim_{n\to\infty} \ln(n) = \lim_{x\to\infty} \ln(x) = \infty$.

Solution：(%i1) limit(log(n),n,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例定義之方程式為 $\ln(n)$，極限變數為 n，範圍為 n 趨近於∞

(%o1) ∞

Example 6.

$\lim_{n\to\infty} c^{1/n} = \lim_{x\to 0^+} c^x = c^0 = 1$, if $c > 0$.

Solution：(%i1) limit(c^(1/n),n,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例定義之方程式為 $c^{1/n}$ ，極限變數為 n，範圍為 n 趨近於∞

(%o1) 1

Example 7.

Evaluate the limits

(a) $\lim_{n\to\infty}(1+\frac{1}{c})^n$ where $c>0$,

(b) $\lim_{n\to\infty}(1+\frac{1}{n})^c$ where $c>0$,

(c) $\lim_{n\to\infty}(1+\frac{1}{n})^n$.

Solution：(%i1) assume (c>0);　假設的指令：assume(假設內容)　//在這裡假設 c>0

(%o1) [c>0]

(%i2) limit((1+1/c)^n,n,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例定義之方程式為 $(1+\frac{1}{c})^n$ ，極限變數為 n，範圍為 n 趨近於∞

(%o2) ∞

(%i3) limit((1+1/n)^c,n,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例定義之方程式為 $(1+\frac{1}{n})^c$ ，極限變數為 n，範圍為 n 趨近於∞

(%o3) 1

(%i4) limit((1+1/n)^n,n,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例定義之方程式為 $(1+\frac{1}{n})^n$ ，極限變數為 n，範圍為 n 趨近於∞

(%o4) %e

The answer are

(a) $\lim_{n\to\infty}(1+\frac{1}{c})^n = \lim_{n\to\infty}(1+\frac{1}{c})^\infty = \infty$.

(b) $\lim_{n\to\infty}(1+\frac{1}{n})^c = \lim_{x\to 0^+}(1+x)^c = 1$.

(c) $\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{x\to\infty}(1+\frac{1}{x})^\infty = e$.

Example 8.

From Theorem 1, the following sequences all approach ∞.

$$\frac{\sqrt{2}}{\ln 2},\frac{\sqrt{3}}{\ln 3},\frac{\sqrt{4}}{\ln 4},\dots,\frac{\sqrt{n}}{\ln(n)},\dots$$

$$\frac{2^1}{1^{10}},\frac{2^2}{2^{10}},\frac{2^3}{3^{10}},\frac{2^4}{4^{10}},\dots,\frac{2^n}{n^{10}},\dots$$

$$\frac{1!}{100^1},\frac{2!}{100^2},\frac{3!}{100^3},\frac{4!}{100^4},\dots,\frac{n!}{100^n},\dots$$

If $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} 1/a_n = 0$ because $1/a_H$ will be infinitesimal.

9.2 Series

Example 1.

$$1+0.1+0.01+0.001+\dots = \frac{1}{1+1/10} = 1\frac{1}{9}.$$

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots = \frac{1}{1-(-1/2)} = \frac{2}{3}.$$

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
 (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1,1/10,inf); 等比數列之和指令：geosum(首項，公比，最末項) // 此題首項爲 1，公比爲 1/10，取到無窮大

(%o2) $\frac{10}{9}$

(%i3) geosum (1,-1/2,inf); 等比數列之和指令：geosum(首項，公比，最末項) // 此題首項爲 1，公比爲-1/2，取到無窮大

(%o3) $\frac{2}{3}$

Example 2.

Every sequence $S_1, S_2, S_3, \ldots, S_n, \ldots$

is the partial sum sequence of an infinite series, namely

$$S_1 + (S_2 - S_1) + (S_3 - S_2) + \ldots + (S_{n+1} - S_n) + \ldots.$$

For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

is the partial sum sequence of

$$1 + (\frac{1}{2} - 1) + (\frac{1}{3} - \frac{1}{2}) + \ldots + (\frac{1}{n+1} - \frac{1}{n}) + \ldots$$

or $1 - \frac{1}{2} - \frac{1}{6} - \ldots - \frac{1}{n(n+1)} - \ldots.$

Example 3.

The harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

is the example promised in our warning. It has the property that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

and yet the series diverges.

Solution：(%i1) load(solve_rec); 因 Maxima 並不會自動做解遞迴式，於是我們必須讀取模組，所謂模組就是 Maxima 內建的一小段程式，在此我們要讀取 solve_rec 這個模組，來做遞迴式的解 //讀取模組 solve_rec

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/solve_rec/

(%i2) solve_rec(a[n+1]=a[n]/(1+a[n]),a[n],a[1]=1); 解遞迴式指令：solve_rec(遞迴式，變數，初始值) //遞迴式由題目推知 $a_{n+1} = a_n /(1+a_n)$ ，變數所求爲 a_n ，初始值爲 1

(%o2) $a_n = \frac{n+1}{n} - 1$

(%i3) ratsimp(%); //將上式做整理得到 $a_n = \frac{1}{n}$

(%o3) $a_n = \frac{1}{n}$

(%i4) limit(%,n,inf); 極限指令：limit(方程式，極限變數，範圍) //此例定義之方程式爲 $\frac{1}{n}$ ，極限變數爲 n，範圍爲 n 趨近於∞

(%o4) $\lim_{n \to \infty} a_n = 0$

9.3 Properties of Infinite Series

Example 1.

For any constant b, and any $|c| < 1$,

$$b + bc + bc^2 + ... + bc^n + ... = b(1 + c + c^2 + ... + c^n + ...)$$

$$= \frac{b}{1-c}.$$

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
  (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) assume(abs(c)<1); 假設的指令：assume(假設內容) //在這裡假設 $|c|<1$

(%o2) $[|c|<1]$

(%i3) geosum(b,c,inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲 b，公比爲 c，取到無窮大

(%o3) $\frac{b}{1-c}$

Example 2.

The series $\frac{1}{5^3}+\frac{1}{5^4}+\frac{1}{5^5}+...+=\sum_{n=3}^{\infty}(\frac{1}{5})^n$

is a tail of the geometric series

$\sum_{n=0}^{\infty}(\frac{1}{5})^n$.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
  (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1/(5^3),1/5,inf); 等比數列之和指令：geosum(首項，公比，最末項)

//此題首項爲 $\frac{1}{5^3}$ ，公比爲 $\frac{1}{5}$ ，取到無窮大

(%o2) $\frac{1}{100}$

Its sum can be found in two ways.

(a) $\sum_{n=3}^{\infty}(\frac{1}{5})^n=\sum_{n=0}^{\infty}(\frac{1}{5})^n-\sum_{n=0}^{2}(\frac{1}{5})^n=\frac{1}{1-\frac{1}{5}}-(1+\frac{1}{5}+\frac{1}{25})=\frac{1}{100}$.

(b) $\sum_{n=3}^{\infty}(\frac{1}{5})^n=\frac{1}{5^3}\sum_{n=0}^{\infty}(\frac{1}{5})^n=\frac{1}{125}\cdot\frac{1}{1-\frac{1}{5}}=\frac{1}{125}\cdot\frac{5}{4}=\frac{1}{100}$.

Example 3.

Here is a convergent geometric series.

$$\frac{1}{5^0}+\frac{1}{5^1}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\frac{1}{5^5}+\dots=\frac{1}{1-\frac{1}{5}}=\frac{5}{4}=1.25.$$

The following series still converges by Corollary 2. Find its sum.

$$3-8+\frac{1}{5^3}+\frac{1}{5^4}+\frac{1}{5^5}+\dots.$$

We have

$$\begin{aligned}3-8+\frac{1}{5^3}+\frac{1}{5^4}+\frac{1}{5^5}+\dots&=3-8+\frac{1}{5^3}(\frac{1}{5^0}+\frac{1}{5^1}+\frac{1}{5^2}+\dots)\\&=(3-8)+\frac{1}{5^3}\cdot\frac{5}{4}=-5+\frac{1}{100}\\&=-4.99.\end{aligned}$$

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求

等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) 3-8+geosum(1/(5^3),1/5,inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲$\frac{1}{5^3}$，公比爲$\frac{1}{5}$，取到無窮大，再+3-8

(%o2) $-\frac{499}{100}$

9.4 Series with Positive Terms

Example 1.

The harmonic series diverges to ∞,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}+...=\infty.$$

This is because it is a positive term series and we have shown that it diverges.

Example 2.

If $0<a$ the geometric series

$$1+a+a^2+...+a^n+...$$

is a positive term series. It converges when $a<1$ and diverges to ∞ when $a\geq 1$.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) assume (a>0); 假設的指令：assume(假設內容) //在這裡假設 $a>0$

```
(%o2) [a>0]
```

(%i3) geosum(1,a,inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲 1，公比爲 a，取到無窮大，結果顯示若 $a<1$，此數列會收斂至 $\frac{1}{1-a}$，反之當 $a\geq 1$ 則此數列會發散至 ∞

$$(\%o3)\ \text{if } a<1 \text{ then } \frac{1}{1-a} \text{ else limit}\left(\frac{1-a^i}{1-a}, i, \infty\right)$$

Example 3.

Test the series $\sum_{n=1}^{\infty} 6^n/(7^n-5^n)$ for convergence. Intuitively, the 7^n should overcome the -5^n, so we shall compare with $6^n/7^n$. The simplest approach is to factor out 7^n. We have

$$\frac{6^n}{7^n-5^n}=\frac{6^n}{7^n(1-(5/7)^n)}\leq\frac{6^n}{7^n(2/7)}=\frac{7}{2}(\frac{6}{7})^n.$$

The geometric series $\sum_{n=1}^{\infty}(6/7)^n$ is convergent, so the given series converges.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(3,(6^n)/(7^n-5^n),inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲 3，公比爲 $\frac{6^n}{7^n-5^n}$，取到無窮大，結果顯示若 $\frac{6^n}{|7^n-5^n|}<1$，

此數列會收斂至$\dfrac{3}{1-\dfrac{6^n}{7^n-5^n}}$，反之當$\dfrac{6^n}{|7^n-5^n|}\geq 1$則此數列會發散至$\infty$

(%o2) if $\dfrac{6^n}{|7^n-5^n|}<1$ then $\dfrac{3}{1-\dfrac{6^n}{7^n-5^n}}$ else

$$\text{limit}\left(\frac{3\left(1-\dfrac{6^{in}}{(7^n-5^n)^i}\right)}{1-\dfrac{6^n}{7^n-5^n}}, i, \infty\right)$$

Example 4.

Test for convergence： $\sum_{n=1}^{\infty} n^2/(n^3+1)$. We have $n^3+1\leq 2n^3$, so

$\dfrac{n^2}{n^3+1}\geq\dfrac{n^2}{2n^3}=\dfrac{1}{2}\cdot\dfrac{1}{n}$.

The harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, whence the given series diverges.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1/2,(n^2)/((n^3)+1),inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲 1/2，公比爲$\dfrac{n^2}{n^3+1}$，取到無窮大，結果顯示若$\dfrac{n^2}{|n^3+1|}<1$，

此數列會收斂至$\dfrac{1}{2(1-\dfrac{n^2}{n^3+1})}$，反之當$\dfrac{n^2}{|n^3+1|}\geq 1$則此數列會發散至$\infty$

$$(\%o2)\quad \text{if } \frac{n^2}{|n^3+1|}<1 \text{ then } \frac{1}{2\left(1-\frac{n^2}{n^3+1}\right)} \text{ else } \operatorname{limit}\left(\frac{1-\frac{|n|^{2i}}{(n^3+1)^i}}{2\left(1-\frac{n^2}{n^3+1}\right)}, i, \infty\right)$$

Example 5.

Test $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^p}$ where p is a positive constant.

We compare this series with the divergent series

$\sum_{n=2}^{\infty}\frac{1}{n}$.

Let H be positive infinite. Then by Theorem 1 in Section 9.1,

$\ln H < H^{1/p}$,

$(\ln H)^p < H$,

$\frac{1}{(\ln H)^p} > \frac{1}{H}$.

By the Limit Comparison Test, the given series $\sum_{n=2}^{\infty} 1/(\ln n)^p$ diverges.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) assume(p>0); 假設的指令：assume(假設內容) //在這裡假設 $p>0$

```
(%o2) [p>0]
```

(%i3) geosum(1/(log(2))^p,1/(log(n))^p,inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲 $\frac{1}{(\ln 2)^p}$，公比爲 $\frac{1}{(\ln n)^p}$，取到無窮大，結果顯示若 $\left|\frac{1}{\ln(n)^p}\right|<1$，此數列會收斂至 $\frac{1}{(\ln 2)^p(1-\frac{1}{\ln(n)^p})}$，反之當 $\left|\frac{1}{\ln(n)^p}\right|\ge 1$ 則此數列會發散至 ∞

$$\text{(\%o3) if } \left|\frac{1}{\log(n)^p}\right|<1 \text{ then } \frac{1}{\log(2)^p\left(1-\frac{1}{\log(n)^p}\right)} \text{ else } \text{limit}\left(\frac{1-\frac{1}{(\log(n)^p)^i}}{\log(2)^p\left(1-\frac{1}{\log(n)^p}\right)}, i, \infty\right)$$

Example 6.

The p series $\sum_{n=1}^{\infty}\frac{1}{n\sqrt[3]{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$ converges because $4/3>1$. The p series $\sum_{n=1}^{\infty}1/\sqrt{n}$ diverges to ∞ because $1/2<1$.

The p series is often used in the Comparison Tests.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
  (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1,1/(n*(n)^(1/3)),inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲 1，公比爲 $\frac{1}{n\sqrt[3]{n}}$，取到無窮大，結果顯示若 $\frac{1}{n^{4/3}}<1$，此

數列會收斂至$\dfrac{1}{1-\dfrac{1}{n^{4/3}}}$，反之當$\dfrac{1}{n^{4/3}}\ge 1$則此數列會發散至$\infty$

$$\text{(\%o2) if } \frac{1}{n^{4/3}}<1 \text{ then } \frac{1}{1-\frac{1}{n^{4/3}}} \text{ else limit}\left(\frac{1-\frac{1}{|n|^{\frac{4i}{3}}}}{1-\frac{1}{n^{4/3}}}, i, \infty\right)$$

Example 7.

Test the series $\sum_{n=1}^{\infty}\dfrac{\ln n}{n^2}$

for convergence.

If H is positive infinite then by Theorem 1 in Section 9.1,

$\ln H < H^c$,

$\dfrac{\ln H}{H^2} < \dfrac{H^c}{H^2} = \dfrac{1}{H^{2-c}}$, for real $c > 0$.

Now take c so that $0 < c < 1$. Then $2-c>1$ so the p series $\sum_{n=1}^{\infty} 1/n^{2-c}$

converges. By the Limit Comparison Test, the given series $\sum_{n=1}^{\infty}(\ln n)/n^2$ converges.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
  (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(log(1),(log(n))/(n^2),inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲 ln(1) ，公比爲$\dfrac{\ln(n)}{n^2}$，取到無窮大，結果顯示若

$\frac{|\log(n)|}{n^2} < 1$，此數列會收斂至 0 ，反之當 $\frac{|\log(n)|}{n^2} \geq 1$ 則此數列會發散至 ∞

(%o2) if $\frac{|\log(n)|}{n^2} < 1$ then 0 else limit(0, i, ∞)

Example 8.

Use the Integral Test to test the improper integral $\int_3^\infty ((\ln x)/x^2)dx$ for convergence.

By Example 7 the series $\sum_{n-=3}^{\infty} (\ln n)/n^2$ converges. For $x > 1$ the function

$f(x) = (\ln x)/x^2$ is continuous, positive, and has derivative

$f'(x) = x^{-3}(1 - 2\ln x)$.

Thus for $x > \sqrt{e}$, $f'(x) < 0$ and $f(x)$ is decreasing . Therefore the Integral Test applies and the improper integral converges.

9.5 Alternating Series

Example 1.

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

converges by the Alternating Series Test, because $\frac{1}{n}$ is decreasing and approaches zero as $n \to \infty$. The partial sums are

$$1, \frac{1}{2}, \frac{5}{6}, \frac{7}{12}, \frac{47}{60}, \frac{37}{60}, \ldots$$

or $\frac{60}{60}, \frac{30}{60}, \frac{50}{60}, \frac{35}{60}, \frac{47}{60}, \frac{37}{60}, \ldots$.

The sum S is between any two consecutive partial sums, for example

$$\frac{37}{60} < S < \frac{47}{60}.$$

Example 2.

The alternating series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \ldots + (-1)^n \frac{n+1}{n} + \ldots$$

diverges. The terms $(n+1)/n$ are decreasing, but their limit is one instead of zero,

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

The Cauchy Test for Divergence in Section 9.2 shows that if the terms a_n do not converge to zero the series diverges.

9.6 Absolute and Conditional Convergence

Example 1.

The alternating series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots,$$

is absolutely convergent, because its absolute value series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

is convergent.

Example 2.

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

is conditionally convergent. It converges by the Alternating Series Test. But its absolute value series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

diverges.

Example 3.

Test the series $\sum_{n=1}^{\infty}\frac{1}{n!}$.

$$\lim_{n\to\infty}\frac{1/(n+1)!}{1/n!}=\lim_{n\to\infty}\frac{1}{n+1}=0,$$

so by the Ratio Test the series converges.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1,1/n!,inf); 等比數列之和指令：geosum(首項，公比，最末項) // 此題首項爲 1，公比爲 $\frac{1}{n!}$，取到無窮大，結果顯示若 $\frac{1}{|n!|}<1$，此數列會收斂至 $\frac{1}{1-\frac{1}{n!}}$，反之當 $\frac{1}{|n!|}\geq 1$則此數列會發散至∞

$$\text{(\%o2) if } \frac{1}{|n!|}<1 \text{ then } \frac{1}{1-\frac{1}{n!}} \text{ else } \operatorname{limit}\left(\frac{1-\frac{1}{n!^{i}}}{1-\frac{1}{n!}}, i, \infty\right)$$

Example 4.

Test $\sum_{n=1}^{\infty}\frac{(-1)^n n^n}{n!}$.

$$\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}\right)=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

e is greater than one, so by the Ratio Test the series diverges.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
  (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(-1,((-1)^n*(n^n))/n!,inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲-1，公比爲 $\frac{(-1)^n n^n}{n!}$，取到無窮大，結果顯示若 $\frac{|n^n||(-1)^n|}{|n!|}<1$，此數列會收斂至 $-\frac{1}{1-\frac{n^n(-1)^n}{n!}}$，反之當 $\frac{|n^n||(-1)^n|}{|n!|}\geq 1$ 則此數列會發散至 ∞

(%o2) $\text{if } \frac{|n^n||(-1)^n|}{|n!|}<1 \text{ then } -\frac{1}{1-\frac{n^n(-1)^n}{n!}} \text{ else}$

$$\text{limit}\left(\frac{\left(\frac{n^n(-1)^n}{n!}\right)^i-1}{1-\frac{n^n(-1)^n}{n!}}, i, \infty\right)$$

Example 5.

The Ratio Test does not apply to either of the series

$$\sum_{n=1}^{\infty}\frac{1}{n}, \quad \sum_{n=1}^{\infty}\frac{1}{n^2},$$

since $\lim_{n\to\infty}\frac{1/(n+1)}{1/n}=1, \quad \lim_{n\to\infty}\frac{1/(n+1)^2}{1/n^2}=1.$

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求

等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1,1/n,inf); 等比數列之和指令：geosum(首項，公比，最末項) //

此題首項爲 1，公比爲$\frac{1}{n}$，取到無窮大，結果顯示若$\frac{1}{|n|}<1$，此數列會收斂至

$\frac{1}{1-\frac{1}{n}}$，反之當$\frac{1}{|n|}\geq 1$則此數列會發散至∞

$$(\%o2)\quad \text{if } \frac{1}{|n|}<1 \text{ then } \frac{1}{1-\frac{1}{n}} \text{ else } \operatorname{limit}\left(\frac{1-\frac{1}{n^i}}{1-\frac{1}{n}}, i, \infty\right)$$

(%i3) geosum(1,1/(n^2),inf); 等比數列之和指令:geosum(首項,公比,最末項) //

此題首項爲 1，公比爲$\frac{1}{n^2}$，取到無窮大，結果顯示若$\frac{1}{n^2}<1$，此數列會收斂至

$\frac{1}{1-\frac{1}{n^2}}$，反之當$\frac{1}{n^2}\geq 1$則此數列會發散至$\infty$

$$(\%o3)\quad \text{if } \frac{1}{n^2}<1 \text{ then } \frac{1}{1-\frac{1}{n^2}} \text{ else } \operatorname{limit}\left(\frac{1-\frac{1}{|n|^{2i}}}{1-\frac{1}{n^2}}, i, \infty\right)$$

9.7 Power Series

Example 1.

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} b^n x^n, \quad \text{where } \; b>0.$$

This just the geometric series

$1 + bx + (bx)^2 + ... + (bx)^n + ...$.

It converges absolutely when $|bx| < 1$, $|x| < 1/b$, and diverges when $|bx| > 1$,

$|x| > 1/b$. So the radius of convergence is $r = 1/b$. At $x = r$ and at $x = -r$ the

series diverges, because $b^n r^n = 1$. Thus the interval of convergence is $(-1/b, 1/b)$.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1,bx,inf); 等比數列之和指令：geosum(首項，公比，最末項) //

此題首項爲 1，公比爲 bx ，取到無窮大，結果顯示若 $|bx| < 1$ ，此數列會收斂至

$\frac{1}{1-bx}$ ，反之當 $|bx| \geq 1$ 則此數列會發散至 ∞

(%o2) if $|bx| < 1$ then $\frac{1}{1-bx}$ else $\mathrm{limit}\left(\frac{1-bx^i}{1-bx}, i, \infty\right)$

Example 2.

Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^n}{n} + ...$$

We compute the limit

$$\lim_{n\to\infty} \frac{|x^{n+1}|/(n+1)}{|x^n/n|} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|.$$

By the Ratio Test the series converges for $|x| < 1$ and diverges for $|x| > 1$, so the

radius of convergence is $r = 1$.

At $x=1$ the series is

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$$

which is divergent. At $x=-1$ the series is

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+...+\frac{(-1)^n}{n}+...$$

which converges by the Alternating Series Test. The interval of convergence is

$[-1,1)$.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(x,(x^n)/n,inf); 等比數列之和指令:geosum(首項,公比,最末項) //

此題首項爲 x，公比爲 $\frac{x^n}{n}$，取到無窮大，結果顯示若 $\frac{|x^n|}{|n|}<1$，此數列會收斂至

$\frac{x}{1-\frac{x^n}{n}}$，反之當 $\frac{|x^n|}{|n|}\geq 1$ 則此數列會發散至 ∞

$$\text{(\%o2)}\quad \text{if } \frac{|x^n|}{|n|}<1 \text{ then } \frac{x}{1-\frac{x^n}{n}} \text{ else } \text{limit}\left(\frac{x\left(1-\left(\frac{x^n}{n}\right)^i\right)}{1-\frac{x^n}{n}}, i, \infty\right)$$

Example 3.

Find the interval of convergence of

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots+\frac{x^n}{n!}+\ldots.$$

For all x we have

$$\lim_{n\to\infty}\frac{\left|x^{n+1}/(n+1)!\right|}{\left|x^n/n!\right|}=\lim_{n\to\infty}\frac{|x|}{n}=0.$$

Therefore by the Ratio Test the series converges for all x. It has radius of convergence ∞, and interval of convergence $(-\infty,\infty)$.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
  (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1,(x^n)/(n!),inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲1，公比爲 $\frac{x^n}{n!}$，取到無窮大，結果顯示若 $\frac{|x^n|}{|n!|}<1$，此數列會收斂至 $\frac{1}{1-\frac{x^n}{n!}}$，反之當 $\frac{|x^n|}{|n!|}\geq 1$則此數列會發散至∞

$$\text{(\%o2)}\quad \text{if } \frac{|x^n|}{|n!|}<1 \text{ then } \frac{1}{1-\frac{x^n}{n!}} \text{ else } \text{limit}\left(\frac{1-\left(\frac{x^n}{n!}\right)^i}{1-\frac{x^n}{n!}}, i, \infty\right)$$

Example 4.

Find the radius of convergence of

$$\sum_{n=0}^{\infty}n!x^n=1+x+2x^2+6x^3+\ldots.$$

For $x \neq 0$, $\lim_{n\to\infty} \frac{|(n+1)!x^{n+1}|}{|n!x^n|} = \lim_{n\to\infty} n|x| = \infty$.

By the Ratio Test the series diverges for $x \neq 0$ and the radius of convergence is

$r = 0$.

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
  (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1,n!*(x^n),inf); 等比數列之和指令：geosum(首項，公比，最末項)

//此題首項爲1，公比爲 $n!x^n$，取到無窮大，結果顯示若 $|n!||x^n| < 1$，此數列會收斂至 $\frac{1}{1-n!x^n}$，反之當 $|n!||x^n| \geq 1$ 則此數列會發散至 ∞

(%o2) $\text{if } |n!||x^n| < 1 \text{ then } \frac{1}{1-n!\,x^n} \text{ else } \text{limit}\left(\frac{1-(n!\,x^n)^i}{1-n!\,x^n}, i, \infty\right)$

Example 5.

Find the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(x+5)^n = 1 + \frac{1}{2}(x+5) + \frac{(2!)^2}{4!}(x+5)^2 + \ldots.$$

We have

$$\lim_{n\to\infty}\left|\frac{(n+1)!(n+1)!(x+5)^{n+1}/(2n+2)!}{(n!)(n!)(x+5)^n/(2n)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(x+5)}{(2n+1)(2n+2)}\right| = \frac{|x+5|}{4}.$$

By the Ratio Test the series converges for $|x+5| < 4$ and diverges for $|x+5| > 4$.

The radius of convergence is $r = 4$, and the interval of convergence is centered at -5.

We note that

$$\frac{(k!)^2}{(2k)!} = \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdot\ldots\cdot\frac{n}{(2n-1)}\cdot\frac{n}{2n} > (\frac{1}{2}\cdot\frac{1}{2})^n = (\frac{1}{4})^n .$$

Therefore at $|x+5| < 4$,

$$\left|\frac{(n!)^2}{(2n)!}(x+5)^n\right| > (\frac{1}{4})^n 4^n = 1.$$

Thus at $x+5=4$ and $x+5=-4$ the terms do not approach zero and the series diverges. The interval of convergence is (-9, -1).

Solution：(%i1) load(functs); 我們想要求等比數列的和，因爲 Maxima 並沒有直接的指令，於是我們必須要讀取 functs 這個模組，用它內建的 geosum 指令來求等比數列的和 //讀取模組 functs

```
Warning - you are redefining the Maxima function lcm
  (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/simplification/fun
```

(%i2) geosum(1,((n!)^2/(2*n)!)*(x+5)^n,inf); 等比數列之和指令：geosum(首項，公比，最末項) //此題首項爲1，公比爲 $\frac{(n!)^2}{(2n)!}(x+5)^n$，取到無窮大，結果顯示若 $\frac{n!^2\left|(x+5)^n\right|}{|(2n)!|} < 1$，此數列會收斂至 $\frac{1}{1-\frac{n!^2(x+5)^n}{(2n)!}}$，反之當 $\frac{n!^2\left|(x+5)^n\right|}{|(2n)!|} \geq 1$ 則此數列會發散至 ∞

$$\text{(\%o2) if } \frac{n!^2\left|(x+5)^n\right|}{|(2n)!|} < 1 \text{ then } \frac{1}{1-\frac{n!^2(x+5)^n}{(2n)!}} \text{ else}$$

$$\text{limit}\left(\frac{1-|n!|^{2i}\left(\frac{(x+5)^n}{(2n)!}\right)^i}{1-\frac{n!^2(x+5)^n}{(2n)!}}, i, \infty\right)$$

9.8 Derivatives and Integrals of Power Series

Example 1.

Differentiate and integrate the power series $\sum_{n=0}^{\infty} n^2 x^n$, and find the radii of convergence.

Solution：(%i1) f(x):=sum((n^2)*(x^n),n,0,inf); //將 $\sum_{n=0}^{\infty} n^2 x^n$ 定為函數 $f(x)$

$$(\%o1)\quad f(x) := \sum_{n=0}^{\infty} n^2 x^n$$

(%i2) integrate(f(x),x,o,inf); //將 $\sum_{n=0}^{\infty} n^2 x^n$ 作積分並從 0 積到 ∞

$$(\%o2)\quad \lim_{x \to \infty-} \left(\sum_{n=0}^{\infty} \frac{n^2 x^{n+1}}{n+1} \right) - \lim_{x \to o+} \sum_{n=0}^{\infty} \frac{n^2 x^{n+1}}{n+1}$$

(%i3) diff(f(x),x); //將 $\sum_{n=0}^{\infty} n^2 x^n$ 對 x 微分

$$(\%o3)\quad \sum_{n=0}^{\infty} n^3 x^{n-1}$$

By the Ratio Test this power series has radius of convergence $r=1$, for

$$\lim_{n\to\infty} \frac{\left|(n+1)^2 x^{n+1}\right|}{\left|n^2 x^n\right|} = |x| \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = |x| .$$

Derivative： $\frac{d}{dx}\left(\sum_{n=0}^{\infty} n^2 x^n\right) = \sum_{n=1}^{\infty} n^3 x^{n-1} = \sum_{m=0}^{\infty} (m+1)^3 x^m$.

For convenience we rewrote the derivative as a power series in x^m where $m=n-1$, and the integral as a power series in x^m where $m=n+1$. Both the derivative and integral also have radius of convergence $r=1$.

9.9 Approximations by Power Series

Example 1.

Approximate $\ln(1\frac{1}{2})$ within 0.01.

Solution：(%i1) float(log(3/2)); //求 $\ln(1\frac{1}{2})$ 的值，且用浮點數表示

(%o1) 0.40546510810816

We use the power series for $\ln(1-x)$,

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\ldots, \quad r=1.$$

Setting $1-x=1\frac{1}{2}$, $x=-\frac{1}{2}$,

$$\ln(1\frac{1}{2})=\frac{1}{2}-\frac{1}{2\cdot 4}+\frac{1}{3\cdot 8}-\frac{1}{4\cdot 16}+\frac{1}{5\cdot 32}-\ldots.$$

This is an alternating series. The last term shown is less than 0.01,

$$\frac{1}{5\cdot 32}=\frac{1}{160}\sim 0.006.$$

By the Alternating Series Test, the error in each partial sum is less than the next term.

So

$$\ln(1\frac{1}{2})\sim\frac{1}{2}-\frac{1}{2\cdot 4}+\frac{1}{3\cdot 8}-\frac{1}{4\cdot 16}, \text{ error}\le\frac{1}{5\cdot 32},$$

or $\ln(1\frac{1}{2})\sim 0.401$, $\text{error}\le 0.006$.

The actual value is $\ln(1\frac{1}{2})\sim 0.405$.

Example 2.

Approximate $\arctan\frac{1}{2}$ within 0.001.

Solution：(%i1) float(atan(1/2)); //求 $\arctan\frac{1}{2}$ 的值，且用浮點數表示

```
(%o1) 0.46364760900081
```

The power series for $\arctan x$ is

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots, \quad r = 1.$$

Setting $x = \frac{1}{2}$,

$$\arctan\frac{1}{2} = \frac{1}{2} - \frac{1}{3\cdot 8} + \frac{1}{5\cdot 32} - \frac{1}{7\cdot 128} + \frac{1}{9\cdot 512} - \ldots.$$

This is an alternating series. The last term is less than 0.001,

$$\frac{1}{9\cdot 512} \sim 0.0002.$$

Therefore

$$\arctan\frac{1}{2} \sim \frac{1}{2} - \frac{1}{3\cdot 8} + \frac{1}{5\cdot 32} - \frac{1}{7\cdot 128}, \quad \text{error} \le 0.0002.$$

Adding up, $\arctan\frac{1}{2} \sim 0.4635$, error ≤ 0.0002.

The series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, \quad r = 1$$

can be used to approximation π. We start with

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, \quad \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

Setting $x = 1/\sqrt{3}$ in the series,

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \frac{1}{7}\left(\frac{1}{\sqrt{3}}\right)^7 + \ldots,$$

or $\frac{\sqrt{3}}{6}\pi = 1 - \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{5}\left(\frac{1}{3}\right)^2 - \frac{1}{7}\left(\frac{1}{3}\right)^3 + \frac{1}{9}\left(\frac{1}{4}\right)^4 - \ldots,$

This is an alternating series, so

$$\frac{\sqrt{3}}{6}\pi \sim 1-\frac{1}{9}+\frac{1}{45}-\frac{1}{189}+\frac{1}{729}, \quad \text{error} \le \frac{1}{11}(\frac{1}{3})^5,$$

$$\frac{\sqrt{3}}{6}\pi \sim 0.9072, \quad \text{error} \le 0.0004.$$

Dividing everything by $\sqrt{3}/6$ we get

$\pi \sim 3.1426, \quad \text{error} \le 0.0013$.

Example 3.

Approximate e^{-1} within 0.001.

The power series for e^x is

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots, \quad r=\infty.$$

Setting $x=-1$,

$$e^{-1} = 1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}-\frac{1}{5040}+\ldots.$$

The series alternates and the last term is less than 0.001, so

$$e^{-1} \sim 1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}, \quad \text{error} \le \frac{1}{5040} \sim 0.0002.$$

Adding up, $e^{-1} \sim 0.36806$, $\text{error} \le 0.0002$.

The actual value is $e^{-1} \sim 0.36788$.

Example 4.

Approximate $1/(1-0.02)$ to six decimal places.

Solution：(%i1) float(1/(1-0.02)); //求 $1/(1-0.02)$ 的值，且用浮點數表示

(%o1) 1.020408163265306

Take $x=0.02$.

$$\frac{1}{1-0.02} = 1 + 0.02 + (0.02)^2 + (0.02)^3 + E_4$$

$$= 1 + 0.02 + 0.0004 + 0.000008 + E_4$$

$$= 1.020408 + E_4 .$$

The error E_4 after four terms is

$$E_4 = \frac{(0.02)^4}{1-0.02} = \frac{0.00000016}{0.98} < \frac{0.00000016}{0.8} = 0.00000020.$$

So $1/(1-0.02) \sim 1.020408$ to six places.

Example 5.

Given a constant c where $-1 < c < 1$, find a simple error estimate for the power series

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n} - \ldots$$

valid for $-1 < c \le 1$.

We start with the equation

$$(1) \quad \frac{1}{1-t} = (1 + t + t^2 + \ldots + t^n) + E_n, \quad E_n = \frac{t^{n+1}}{1-t}.$$

For $-1 < t \le c$ we have

$$1 - t \ge 1 - c, \quad |E_n| \le \frac{|t|^{n+1}}{1-c}.$$

Integrating Equation 1 from 0 to x we have

$$(2) \quad -\ln(1-x) = (x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^{n+1}}{n+1}) + \int_0^x E_n\,dt$$

and $\left|\int_0^x E_n\,dt\right| \le \int_0^x \frac{|t|^{n+1}}{1-c}dt = \frac{|x|^{n+2}}{(1-c)(n+2)}.$

Multiplying Equation 2 by -1 and setting $m = n+1$ we have the following error estimate for $\ln(1-x)$, valid for $-1 < x \le c$.

$$(3) \quad \ln(1-x) \sim (-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^m}{m}),$$

$$\text{error} \le \frac{|x|^{m+1}}{(1-c)(m+1)}.$$

Example 6.

Use Example 5 to approximate $\ln 1\frac{1}{2}$ within 0.01. We set $c = x = \frac{1}{2}$ in Equation 3.

$$\ln(\frac{1}{2}) \sim -\frac{1}{2} - \frac{1}{2\cdot 4} - \frac{1}{3\cdot 8} - \frac{1}{4\cdot 16} - \ldots - \frac{1}{m\cdot 2^m},$$

$$|error| \le \frac{(1/2)^{m+1}}{\frac{1}{2}(m+1)} = \frac{1}{(m+1)2^m}.$$

Table 9.9.1 shows approximate values and error estimates.

m	$\frac{1}{m\cdot 2^m}$	Approximate value for $\ln\frac{1}{2}$ $-\frac{1}{2} - \frac{1}{2\cdot 4} - \ldots - \frac{1}{m\cdot 2^m}$	Error Estimate $\frac{1}{(m+1)2^m}$
1	0.5000	-0.5000	0.2500
2	0.1250	-0.6250	0.0833
3	0.04167	-0.6667	0.0313
4	0.01563	-0.6823	0.0125
5	0.00625	-0.6886	0.0052

We see that the error estimate drops below 0.01 when $m = 5$.

So $\ln\frac{1}{2} \sim -0.689$, $\text{error} \le 0.01$.

Since $\ln\frac{1}{2} = -\ln 2$, we have

$\ln 2 \sim 0.689$, $\text{error} \le 0.01$.

Example 7.

Find an error estimate for the power series for $\ln((1+x)/(1-x))$ valid for

$-c \le x \le c$. Use it to approximate $\ln 2$ within 0.00001.

From Example 5 we have the following error estimates for $\ln(1+x)$ and $-\ln(1-x)$ valid for $-c \le x \le c$.

$$\ln(1+x) \sim x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{m+1}\frac{x^m}{m},$$

$$\text{error} \le \frac{|x|^{m+1}}{(1-c)(m+1)}.$$

$$-\ln(1-x) \sim x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^m}{m},$$

$$\text{error} \le \frac{|x|^{m+1}}{(1-c)(m+1)}.$$

We add the two sums and error estimates,

$$\ln(\frac{1+x}{1-x}) \sim 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + ... + \frac{2x^{2m-1}}{2m-1},$$

$$\text{error} \le \frac{2|x|^{2m+1}}{(1-c)(2m+1)}.$$

We wish to choose x so that $(1+x)/(1-x) = 2$. Solving for x we get $x = \frac{1}{3}$.

Now set $c = \frac{1}{3}$ and $x = \frac{1}{3}$. The error estimate for $x = \frac{1}{3}$ is

$$\frac{2|x|^{2m+1}}{(1-c)(2m+1)} = \frac{1}{(2m+1)3^{2m}}.$$

Table 9.9.2

m	$\frac{2}{(2m-1)3^{2m-1}}$	Approximate value for $\ln 2$ $\frac{2}{1\cdot 3} + \frac{2}{3\cdot 27} + ... + \frac{2}{(2m-1)3^{2m-1}}$	Error estimate $\frac{1}{(2m+1)3^{2m}}$
1	0.666667	0.666667	0.037037
2	0.024691	0.691358	0.002469
3	0.001646	0.693004	0.000196

4	0.000131	0.693134	0.000017
5	0.000011	0.693146	0.000002

The error estimate drops below 0.00001 when $m = 5$. Thus

$\ln 2 \sim 0.693146$, error ≤ 0.00001.

Example 8.

Find the sum of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

Our first guess is to set $x = -1$ in the power series

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots, \quad r = 1.$$

This guess to us the sum

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

We know the series converges to something by the Alternating Series Test. For $-1 < x < 1$ the series converges to $\ln(1-x)$. But $x = -1$ is an endpoint of the interval of convergence and the general theorem on integrating a power series does not apply. So we must go back to the beginning and use the equation

$$\frac{1}{1-t} = (1 + t + \ldots + t^n) + \frac{t^{n+1}}{1-t}.$$

For $t \leq 0, \left|t^{n+1}/(1-t)\right| \leq \left|t^{n+1}\right|,$ whence

$$\frac{1}{1-t} = (1 + t + \ldots + t^n) + E_n, \quad |E_n| \leq \left|t^{n+1}\right|.$$

Integrating from 0 to x,

$$-\ln(1-x) = (x + \frac{x^2}{2} + \ldots + \frac{x^{n+1}}{n+1}) + F_n, \quad |F_n| \leq \left|\frac{x^{n+2}}{n+2}\right|.$$

This holds for all $x \leq 0$.

Now we set $x=-1$ and see that the error term $|F_n| \le 1/(n+2)$ approaches zero.

This proves that $\ln 2$ really is the sum of the alternating harmonic series,

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots .$$

The alternating harmonic series converges very slowly, because after n terms the error estimate is only $1/(n+1)$.

9.10 Taylor's Formula

Example 1.

Find the first five Taylor polynomials of $\sin x$ at $x=0$. We work them out in Table 9.10.1.

Table 9.10.1

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$P_k(x)$
0	$\sin x$	0	0
1	$\cos x$	1	x
2	$-\sin x$	0	x
3	$-\cos x$	-1	$x - x^3/3!$
4	$\sin x$	0	$x - x^3/3!$
5	$\cos x$	1	$x - x^3/3! + x^5/5!$

Since the even degree terms are zero, the $2n$th Taylor polynomial is the same as the $(2n-1)$st. Figure 9.10.2 compares the first and third Taylor polynomials with $\sin x$.

Example 2.

Find MacLaurin's Formula for $f(x) = e^x$.

The nth derivative is $f^{(n)}(x) = e^x \cdot f^{(n)}(0) = 1$. MacLaurin's Formula is

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + R_n(x)$. $R_n(x) = e^{t_n} \frac{x^{n+1}}{(n+1)!}$ for some t_n between 0 and x. For t between 0 and x the value of e^t is always less than or equal to $3^{|x|}$, for

$e^t \le e^{|x|} \le 3^{|x|}$.

We therefore have the formula

$$(3) \quad e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + R_n(x), \quad |R_n(x)| \le 3^{|x|} \cdot \frac{|x|^{n+1}}{(n+1)!}.$$

The formula (3) can be used to approximate e^x. Let us set $x = 1$ and approximate e. The error estimate is now

$$3^{|x|} \cdot \frac{|x|^{n+1}}{(n+1)!} = \frac{3}{(n+1)!}.$$

n	$1/n!$	Approximate value for e $1+1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}$	Error estimate $\frac{3}{(n+1)!}$
2	0.500000	2.500000	0.500000
3	0.166667	2.666667	0.125000
4	0.041667	2.708333	0.025000
5	0.008333	2.716667	0.004167
6	0.001389	2.718056	0.000594
7	0.000198	2.718254	0.000075
8	0.000025		

This compares with $e = 2.718282$.

Example 3.

Find MacLaurin's Formula for $f(x) = \sin x$. The derivatives are

$f(x) = \sin x \qquad\qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f^{(3)}(x) = -\cos x \qquad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$

. .

. .

. .

MacLaurin's Formula for $2n$ terms is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + R_{2n}(x),$$

$$R_{2n}(x) = (-1)^n \cos t \frac{x^{2n+1}}{(2n+1)!}.$$

For all t, $|\cos t| \le 1$, so we have the error estimate

$$|R_{2n}(x)| \le \frac{|x|^{2n+1}}{(2n+1)!}.$$

9.11 Taylor Series

Example 1.

Let $f(x)$ be a polynomial in $x - c$,

$$f(x) = a_0 + a_1(x-c) + \ldots + a_n(x-c)^n.$$

This is just a power series with all but the first $n+1$ coefficients equal to zero. So by Theorem 1, the Taylor series of the polynomial is just the polynomial itself followed by infinitely many zeros,

$$a_0 + a_1(x-c) + \ldots + a_n(x-c)^n + 0 + 0 + \ldots.$$

We can also see this directly from Lemma 1 of the last section, namely

$$\frac{f^{(m)}(c)}{m!} = a_m \quad \text{for} \quad m \le n.$$

Example 6.

Find the sixth derivative of $f(x) = 1/(1+x^2)$ at $x = 0$.

If we try to differentiate directly we will be hopelessly bogged down at about the third derivative. But from the MacLaurin series we see that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots,$$

$$\frac{f^{(6)}(0)}{6!} x^6 = -x^6,$$

$$\frac{f^{(6)}(0)}{6!} = -1,$$

$$f^{(6)}(0) = -6! = -720.$$

Example 3.

Find the power series for $\arcsin x$.

Solution：(%i1) taylor(asin(x),x,0,10); Taylor 多項式的展開指令：taylor(函數式，變數，第 1 項，第 n 項) //將 $\arcsin x$ 用泰勒展開，展開到第 10 項

(%o1) $x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152} + \ldots$

Recall that for $|x| < 1$,

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x (1-t^2)^{-1/2} dt.$$

We start with the binomial series with $p = -\frac{1}{2}$ and obtain the following power series by substitution and integration. They are valid for $|x| < 1$.

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + (-\frac{1}{2})(-\frac{3}{2})\frac{1}{2!}x^2 - \ldots + (-1)^n \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n n!}x^n + \ldots$$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots + \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n n!}x^n + \ldots$$

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \ldots + \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n n!}x^{2n} + \ldots$$

$$\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots + \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n n!(2n+1)}x^{2n+1} + \ldots .$$