8.1 EXPONENTIAL FUNCTIONS

Example 1 Approximate $\sqrt{2^{\pi}}$. We have

$$\sqrt{2} \sim 1.4142, \qquad \pi \sim 3.14.$$

Thus $$1.414 < \sqrt{2} < 1.415, \qquad 3.1 < \pi < 3.2.$$

By the inequalities for exponents,

$$(1.414)^{3.1} < \sqrt{2^{\pi}} < (1.415)^{3.2},$$

or $$2.91 < \sqrt{2^{\pi}} < 3.06.$$

Thus $\sqrt{2^{\pi}}$ is within $\frac{1}{10}$ of 3.0.

```
(%i1) sqrt(2^%pi);
(%o1) 2^(%pi/2)

(%i2) float(%), numer;
(%o2) 2.970686423552019
```

*

Example 2 If a>1, evaluate the limit $\lim_{x\to\infty} a^x$.

Let H be positive infinite and a=b+1. Then b>0 and by inequality,

$$a^H = (b+1)^H \geq bH + 1.$$

So a^H is positive infinite. Therefore

$$\lim_{x\to\infty} a^x = \infty$$

```
(%i6) assume(a>1);
(%o6) [a>1]
```

(%i8) 'limit(a^x,x,inf);

(%o8) $\lim_{x \to \infty} a^x$

(%i9) limit(a^x,x,inf);

(%o9) ∞

*

Example 3 Evaluate the limit $\lim_{x\to\infty} \dfrac{4^{x+1}+5}{4^{x-1}-3}$.

Let H be positive infinite. Then

$$\frac{4^{H+1}+5}{4^{H-1}-3} = \frac{4^{H+1}\cdot 4^{-H}+5\cdot 4^{-H}}{4^{H-1}\cdot 4^{-H}-3\cdot 4^{-H}} = \frac{4+5\cdot 4^{-H}}{\frac{1}{4}-3\cdot 4^{-H}}$$

By Example 2, 4^H is infinite, so $(\frac{1}{4})^H$ is infinitesimal. Thus

$$st(\frac{4^{H+1}+5}{4^{H-1}-3}) = st(\frac{4+5\cdot 4^{-H}}{\frac{1}{4}-3\cdot 4^{-H}}) = \frac{4+5\cdot 0}{\frac{1}{4}-3\cdot 0} = 16,$$

$$\lim_{x\to\infty} \frac{4^{x+1}+5}{4^{x-1}-3} = 16.$$

(%i12) limit((4^(x+1)+5)/(4^(x-1)-3));

(%o12) $\dfrac{4^{x+1}}{4^{x-1}-3}+\dfrac{5}{4^{x-1}-3}$

(%i13) limit((4^(x+1)+5)/(4^(x-1)-3),x,inf);

(%o13) 16

*

8.2 LOGARITHMIC FUNCTIONS

Example 1 Simplify the term $\log_a(\log_a(a^{a^x}))$.

$$\log_a(\log_a(a^{a^x})) = \log_a(a^x \log_a a) = \log_a(a^x) = x.$$

(%i16) loga(x) := log(x) / log(a);

$$\text{(\%o16)}\quad \text{loga}(x) := \frac{\log(x)}{\log(a)}$$

(%i17) loga(loga(a^a^x));

(%o17) x

Example 2 Express $\log_b\left(\frac{x^3\sqrt{y}}{z}\right)$ in terms of $\log_b x$, $\log_b y$, and $\log_b z$.

$$\log_b\left(\frac{x^3\sqrt{y}}{z}\right) = 3\log_b x + \frac{1}{2}\log_b y - \log_b z.$$

Example 3 Solve the equation below for x.

$$3^{x^2-2x} = \frac{1}{3}$$

We take $\log_3$ of both sides of the equation.

$$(x^2-2x)\log_3 3 = \log_3(3^{-1}),$$

$$x^2-2x = -1,$$
$$x^2-2x+1 = 0,$$
$$x = 1.$$

The inequalities for exponents can be used to compute limits of logarithms.

http://www.npue.edu.tw/academic/math/index.htm

(%i24) solve[3^(x^2-2*x)=1/3];

(%o24) $solve_{3^{x^2-2x}=\frac{1}{3}}$

(%i23) solve([3^(x^2-2*x)=1/3],[x]);

(%o23) $[x=1]$

Example 4 Evaluate the limit $\lim_{x\to\infty}\log_a x$, $a>1$.

Let H be positive infinite. Then $0=\log_a 1<\log_a H$, so $\log_a H$ is positive. If $\log_a H$ is finite, say $\log_a H<n$, then

$$H=a^{\log_a H}<a^n,$$

which is impossible because H is infinite. Therefore $\log_a H$ is positive infinite, so

$$\lim_{x\to\infty}\log_a x=\infty$$

(%i6) assume(a>1);

(%o6) $[a>1]$

(%i16) loga(x) := log(x) / log(a);

(%o16) $\mathrm{loga}(x):=\frac{\log(x)}{\log(a)}$

(%i26) limit(loga(x),x,inf);

(%o26) ∞

8.3 DERIVATIVE OF EXPONENTIAL FUNCTION AND THE NUMBER e

Example 1 Given $y = e^{\sin x}$, find d^2y/dx^2.

$$\frac{dy}{dx} = e^{\sin x}\cos x,$$

$$\frac{d^2y}{dx^2} = e^{\sin x}\cos^2 x - e^{\sin x}\sin x.$$

(%i35) diff(%e^sin(x),x,1);

(%o35) $\cos(x)\,\%e^{\sin(x)}$

(%i34) diff(%e^((sin((x))))*cos((x)),x,1);

(%o34) $\cos(x)^2\,\%e^{\sin(x)} - \sin(x)\,\%e^{\sin(x)}$

(%i32) diff(%e^((sin((x)))),x,2);

(%o32) $\cos(x)^2\,\%e^{\sin(x)} - \sin(x)\,\%e^{\sin(x)}$

Example 2 Find the area under the curve

$$y = \frac{e^{\arctan x}}{1+x^2}, \qquad 0 \le x \le 1.$$

Let $u = \arctan x$, $du = \dfrac{1}{1+x^2}dx.$

Then $$\int_0^1 \frac{e^{\arctan x}}{1+x^2}dx = \int_0^{\pi/4} e^u\,du = e^u\Big]_0^{\pi/4} = e^{\pi/4} - 1.$$

(%i38) integrate(%e^((arctan((x))))/(1+x^2), x, 0, 1);

(%o38) $$\int_0^1 \frac{\%e^{\arctan(x)}}{x^2+1}dx$$

(%i39) integrate(%e^((atan((x))))/(1+x^2), x, 0, 1);

(%o39) $\%e^{\pi/4} - 1$

Example 3 Find $d(a^x)/dx$. We use the formula

$$a = e^{\log_e a}, \quad a^x = e^{x\log_e a}.$$

Put $u = x\log_e a$. Then $a^x = e^u$, so

$$\frac{d(a^x)}{dx} = (\log_e a)a^x.$$

(%i2) log%e(x):=log(x)/log(%e);

(%o2) $\mathrm{log\%e}(x):=\frac{\log(x)}{\log(\%e)}$

(%i3) log%e(a);

(%o3) log(a)

(%i4) a:%e^(log%e(a));

(%o4) a

(%i5) diff(a^x,x,1);

(%o5) $a^x \log(a)$

8.4 SOME USES OF EXPONENTIAL FUNCTIONS

Example 1 Find the area of the region under the catenary $y = \cosh x$ from $x = -1$ to $x = 1$,

$$A = \int_{-1}^{1} \cosh x\,dx = \sinh x\Big]_{-1}^{1}$$
$$= \sinh 1 - \sinh(-1)$$
$$= \frac{e - e^{-1}}{2} - \frac{e^{-1} - e}{2} = e - \frac{1}{e}.$$

Example 2 If money is received at the rate $f(t) = 2t$ dollars per year, and earns interest at the annual rate of 7%, how much will be accumulated from times $t = 0$ to $t = 10$?

The formula gives

$$C = \int_{0}^{10} 2te^{0.07(10-t)}\,dt$$

We first find the indefinite integral.

$$\int 2te^{0.07(10-t)}\,dt = \int 2te^{0.7}e^{-0.07t}\,dt$$
$$= 2e^{0.7}\int te^{-0.07t}\,dt$$

Let $u = -0.07t$, $du = -0.07dt$. Then

$$\int 2te^{0.07(10-t)}\,dt = 2e^{0.7}\int \frac{u}{-0.07}e^{u}\frac{1}{-0.07}\,du$$
$$= 2e^{0.7}(0.07)^{-2}\int ue^{u}\,du$$

Using integration by parts,

$$\int ue^{u}\,du = ue^{u} - \int e^{u}\,du = ue^{u} - e^{u} + \text{Constant}.$$

Therefore $$\int 2te^{0.07(10-t)}\,dt = 2e^{0.7}(0.07)^{-2}(ue^{u} - e^{u}) + \text{Constant}.$$

When $t = 0$, $u = 0$ and when $t = 10$, $u = -0.7$. Thus

$$C = [2e^{0.7}(0.07)^{-2}(ue^{u} - e^{u})]_{0}^{-0.7}$$
$$= 2e^{0.7}(0.07)^{-2}(-0.7e^{-0.7} - e^{-0.7} + e^{0})$$
$$= 2(0.07)^{-2}(e^{0.7} - 1.7) \sim 128.08.$$

The answer is \$128.08.

(%i9) integrate(2*t*%e^((0.07*((10-t)))), t, 0, 10);

```
rat: replaced 0.07 by 7/100 = 0.07
rat: replaced 0.07 by 7/100 = 0.07
rat: replaced 0.07 by 7/100 = 0.07
rat: replaced -0.07 by -7/100 = -0.07
rat: replaced 0.7 by 7/10 = 0.7
rat: replaced 2.013752707470477 by 14789/7344 = 2.013752723311547
```

(%o9) $2\left(\frac{9243125}{22491}-\frac{1848625\,\%e^{-\frac{7}{10}}}{2646}\right)$

(%i10) float(%), numer;

(%o10) 128.0623305871812

8.5 NATURAL LOGARITHMS

Example 1 Find $\frac{d(\log_{10} x)}{dx}$

Right: $\frac{d(\log_{10} x)}{dx} \sim \frac{d(0.4343 \ln x)}{dx} = \frac{0.4343}{x}$

Wrong: $\frac{d(\log_{10} x)}{dx} = \frac{1}{x}$.

(%i13) log10(x):=log(x)/log(10);

(%o13) $\mathrm{log10}(x):=\frac{\log(x)}{\log(10)}$

(%i14) diff(log10((x)),x,1);

(%o14) $\frac{1}{\log(10)\, x}$

(%i15) float(%), numer;

(%o15) $\frac{0.43429448190325}{x}$

Example 2 Find $\int_1^{10} \frac{1}{x} dx$.

Right: $\int_1^{10} \frac{1}{x} dx = \ln x \big]_1^{10} = \ln 10 - \ln 1 \sim 2.3026$

Wrong: $\int_1^{10} \frac{1}{x} dx = \log_{10} x \big]_1^{10} = \log_{10} 10 - \log_{10} 1 = 1$

(%i16) integrate(1/x, x, 1, 10);

(%o16) log(10)

(%i17) float(%), numer;

(%o17) 2.302585092994046

Example 3 Find $\int_{-e}^{-1} \frac{1}{x} dx$

$$\int_{-e}^{-1} \frac{1}{x} dx = \ln|x| \Big]_{-e}^{-1} = \ln 1 - \ln e = -1$$

Note that $\ln x$ is undefined at -1 and $-e$ but $\ln|x|$ is defined there.

The absolute value sign is put in when integrating $1/x$ and removed when differentiating $\ln|x|$.

(%i18) integrate(1/x, x, -%e, -1);

(%o18) -1

Example 4 Find dy/dx where $y = \ln[(3-2x)^2]$.

We have $(3-2x)^2 = |3-2x|^2$, and by the rules of logarithms,

$$y = 2\ln|3-2x|.$$

By Theorem 1, $\frac{dy}{dx} = \frac{2}{3-2x} \frac{d(3-2x)}{dx} = \frac{-4}{3-2x}$

This answer is correct when $3-2x$ is negative as well as positive.

Example 5 Find $d(\log_a x)/dx$

$$\log_a x = \frac{\ln x}{\ln a},$$

$$\frac{d(\log_a x)}{dx} = \frac{1}{\ln a} \frac{d(\ln x)}{dx} = \frac{1}{x \ln a}$$

(%i1) loga(x):=log(x)/log(a);

(%o1) $\mathrm{loga}(x) := \frac{\log(x)}{\log(a)}$

(%i2) diff(loga((x)),x,1);

(%o2) $\frac{1}{\log(a)\, x}$

Example 6 Find $\int \frac{1}{2x-5}dx$. Let $u = 2x-5$, $du = 2dx$.

$$\int \frac{1}{2x-5}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x-5| + C.$$

(%i4) integrate(1/((2*x-5)), x);

(%o4) $\frac{\log(2x-5)}{2}$

Example 7 Find the improper integral $\int_1^\infty \frac{1}{x}dx$.

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx = \lim_{b\to\infty}\left(\ln x]_1^b\right) = \lim_{b\to\infty}\ln b = \infty.$$

Thus the region under the curve $y = 1/x$ from 1 to ∞

(%i5) integrate(1/x, x, 1, inf);

```
defint: integral is divergent.
 -- an error.  To debug this try debugmode(true);
```

(%i7) limit(integrate(1/x, x, 1, b),b,inf);

Is b-1 positive, negative, or zero?positive;

(%o7) ∞

Example 8 The region R under the curve $y = 1/x$ from 1 to ∞ is rotated about the $x-$axis, forming a solid of revolution. Find the volume of this solid

The volume is given by the improper integral

$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^\infty x^{-2} dx$$

Then $$V = \pi \lim_{b\to\infty} \int_1^b x^{-2} dx = \pi \lim_{b\to\infty}\left(\left[-\frac{1}{x}\right]_1^b \right) = \pi \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = \pi$$

Thus the solid has volume π .

```
(%i8) integrate(%pi*((1/x))^2, x, 1, inf);
(%o8) π
```

```
(%i9) limit(%pi*integrate((1/x)^2, x, 1, b),b,inf);
Is  b-1  positive, negative, or zero?positive;
(%o9) π
```

8.6 SOME DIFFERENTIAL EQUATIONS

Example 1 A country has a population of ten million at time $t = 0$, and constant annual birth rate $b = 0.020$ and death rate $d = 0.015$ per person. Find the population at time t.

The population satisfies the differential equation

$$\frac{dy}{dt} = (b-d)y = 0.005y.$$

The initial condition is

$$y = 10^7 \quad \text{at} \quad t = 0.$$

The general solution is

$$y = Ce^{0.005t}.$$

Since at $t = 0$, $10^7 = Ce^0 = C$, the actual solution is

$$y = 10^7 e^{0.005t}.$$

(%i1) 'diff(y,t)-0.005*y=0;

$$\text{(\%o1)} \quad \frac{d}{dt} y - 0.005\, y = 0$$

先整理並列式

(%i2) ode2(%,y,t);

```
rat: replaced -0.005 by -1/200 = -0.005
rat: replaced -0.005 by -1/200 = -0.005
rat: replaced -0.005 by -1/200 = -0.005
```

$$\text{(\%o2)} \quad y = \%c\, \%e^{t/200}$$

解常微分,指令為 ode2 (前式 ,函數 ,變數);

(%i3) ic1(%,t=0,y=10^7);

$$\text{(\%o3)} \quad y = 10000000\, \%e^{t/200}$$

帶入初始值,指令為 ic1 (前式 ,變數的初始值 ,函數的數值);

Example 2 A radioactive element has a half-life of N years, that is, half of the substance will decay every N years. Given ten pounds of the element at time $t=0$, how much will remain at time t?

In radioactive decay the amount y of the element is decreasing at a rate proportional to y, so the differential equation has the form

$$dy/dt = ky.$$

The general solution is

$$y = Ce^{kt}.$$

Since y is decreasing, k will be negative. We must find the constants C and k. To find C we use the initial condition

$$y=10 \quad \text{at} \quad t=0, \qquad C=10.$$

To find k we use the given half-life. It tells us that

$$y=\frac{1}{2}\cdot 10=5 \quad \text{at} \quad t=N.$$

Therefore $$10e^{kN}=5,$$

$$e^{k}=\left(\frac{1}{2}\right)^{1/N},$$

$$k=\ln\left(\left(\frac{1}{2}\right)^{1/N}\right)=-\frac{\ln 2}{N}$$

The solution is $$y=10e^{-(t\ln 2)/N}$$

(%i1) 'diff(y,t)-k*y=0;

(%o1) $\frac{d}{dt}y-k\,y=0$

先整理並列式

(%i2) ode2(%,y,t);

(%o2) $y=\%c\,\%e^{k\,t}$

解常微分,指令爲 ode2 (前式 ,函數 ,變數);

(%i3) ic1(%,t=0,y=10);

(%o3) $y=10\ \%e^{k\,t}$

帶入初始值,指令爲 ic1 (前式 ,變數的初始值 ,函數的數值);

(%i8) k:solve(%,k);

(%o8) $[k=\frac{\log\left(\frac{y}{10}\right)}{t}]$

定義 k 等於上式解出的 k 值

(%i9) k:subst([y=5,t=N],k);

(%o9) $[k=-\frac{\log(2)}{N}]$

將 y 和 t 用 5 和 N 取代並帶入 k,指令爲 subst ([y 值, t 值] , 敘述式);

(%i16) y:10*(%e^(k*t));

(%o16) $10\ \%e^{-\frac{\log(2)\,t}{N}}$

將上式所得到的 k 值代入%o3 式,即可得到 y 解

Example 3 Solve $dy/dx = e^{y}\sin x$

$$e^{-y}dy = \sin x\,dx,$$
$$-e^{-y} = -\cos x - C,$$
$$e^{-y} = \cos x + C,$$
$$-y = \ln(\cos x + C),$$

$$y = -\ln(\cos x + C).$$

Second order differential equations also arise frequently in applications. As a rule, the general solution of a second order differential equation will involve two constants, and two initial conditions are needed to determine a particular solution.

(%i1) 'diff(y,x)-(%e^y)*sin(x)=0;

(%o1) $\frac{d}{d\,x}\,y - \sin(x)\,\%e^{y} = 0$

先整理並列式

(%i2) ode2(%,y,x);

(%o2) $-\%e^{-y} = \%c - \cos(x)$

解常微分,指令為 ode2 (前式 ,函數 ,變數);

(%i3) solve(%,y);

(%o3) $[y = \log\left(-\frac{1}{\%c - \cos(x)}\right)]$

對前式解 y 值

Example 4 Newton's law, $F = ma$, states that force equals mass times acceleration. Suppose a constant force F is applied along the $y-$axis to an object of constant mass m. Then the position y of the object is governed by the second order differential equation

$$m\frac{d^2y}{dt^2} = F, \qquad \frac{d^2y}{dt^2} = \frac{F}{m}.$$

The general solution of this equation is found by integration twice,

$$\frac{dy}{dt} = \frac{Ft}{m} + v_0,$$

$$y = \frac{Ft^2}{2m} + v_0 + y_{0.}$$

Setting $t = 0$ we see that the constants v_0 and y_0 are just the velocity and position at time $t = 0$. Thus the motion of the object is known if we know its initial position y_0 and velocity v_0.

If the force $F(t)$ varies with time we have the differential equation

$$\frac{d^2y}{dt^2} = \frac{F(t)}{m}$$

The general solution can still be found by integrating twice, and the motion will still be determined by the initial position and velocity. Suppose for example that

$F(t) = t^2$, and $y_0 = 5$, $v_0 = 1$ at time $t = 0$. Then

$$\frac{d^2y}{dt^2} = \frac{t^2}{m},$$

$$\frac{dy}{dt} = \frac{t^3}{3m} + 1,$$

$$y = \frac{t^4}{12m} + t + 5.$$

(%i4) m*'diff(y,t,2)-F=0;

(%o4) $m\left(\frac{d^2}{d\,t^2}\,y\right) - F = 0$

(%i5) ode2(%,y,t);

(%o5) $y = \frac{t^2\,F}{2\,m} + \%k2\,t + \%k1$

(%i6) y:%;

(%o6) $y = \frac{t^2\,F}{2\,m} + \%k2\,t + \%k1$

```
(%i7) subst([F=t^2,%k1=5,%k2=1],y);
```

(%o7) $y=\frac{t^4}{2\,m}+t+5$

Example 5. When a spring of natural length L is compressed a distance x it exerts a force $F=-kx$. The negative sign indicates that the force is in the opposite direction from x.

When x is negative the spring is expanded and the equation $F=-kx$ still holds.

Suppose a mass m is attached to the end of the spring and at time $t=0$ is at position x_0 and has velocity v_0. The motion of the mass follows the differential equation

$$F=ma, \qquad -kx=m\frac{d^2x}{dt^2}, \qquad \frac{d^2x}{dt^2}=-\frac{k}{m}x.$$

The general solution is

$$x=a\cos\omega t+b\sin\omega t$$

where $\omega=\sqrt{k/m}.$ Using the initial conditions, the motion of the mass is

$$x=x_0\cos\omega t+\frac{v_0}{\omega}\sin\omega t.$$

This function is periodic with period $2\pi/\omega$, so as expected the mass oscillates back and forth.

8.7 DERIVATIVES AND INTEGRALS INVOLVING lnx

Example 1 Find dy/dx where $y=(2x+1)(3x-1)(4-x)$.

$$\ln|y|=\ln|2x+1|+\ln|3x-1|+\ln|4-x|,$$

$$\frac{dy}{dx}=y(\frac{2}{2x+1}+\frac{3}{3x-1}-\frac{1}{4-x})$$

$$=(2x+1)(3x-1)(4-x)(\frac{2}{2x+1}+\frac{3}{3x-1}-\frac{1}{4-x}).$$

Example 2 Find dy/dx where $y=x^x$.

$$\ln y=x\ln x.$$

$$\frac{dy}{dx}=y\frac{d(x\ln x)}{dx}=x^x(\frac{x}{x}+\ln x)=x^x(1+\ln x).$$

In this example, $\ln y=\ln|y|$ because $y>0$.

Example 3 Find dy/dx where $y=\dfrac{(x^2+1)^3(x^3+x+2)}{(x-1)\sqrt{x+4}}$.

$$\ln|y|=3\ln|x^2+1|+\ln|x^3+x+2|-\ln|x-1|-\frac{1}{2}\ln|x+4|,$$

$$\frac{dy}{dx}=y\left(\frac{6x}{x^2+1}+\frac{3x^2+1}{x^3+x+2}-\frac{1}{x-1}-\frac{1}{2(x+4)}\right)$$

$$=\frac{(x^2+1)^3(x^3+x+2)}{(x-1)\sqrt{x+4}}\left(\frac{6x}{x^2+1}+\frac{3x^2+1}{x^3+x+2}-\frac{1}{x-1}-\frac{1}{2(x+4)}\right).$$

This derivative could have been found using the Product and Quotient Rules but it would take a great deal of work.

Example 4 Find $\int\tan\theta\,d\theta$. We have $\tan\theta=(\sin\theta/\cos\theta)$. Let $u=\cos\theta,\ du=-\sin\theta\,d\theta$. Then

$$\int\tan\theta\,d\theta=-\int 1/u\,du=-\ln|u|+C=-\ln|\cos\theta|+C.$$

Remember the absolute value sign inside the logarithm. It is needed because

$\cos\theta$ may be negative.

Example 5 Find $\int \sec\theta \, d\theta$.

$$\int \sec\theta \, d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta$$

$$= \int \frac{d(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} = \ln|\sec\theta + \tan\theta| + C.$$

Example 6 Find $\int \sec^3\theta \, d\theta$. From the reduction formula in Section 7.5.

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\sec^2\theta \sin\theta + \frac{1}{2}\int \sec\theta \, d\theta .$$

Therefore $\int \sec^3\theta \, d\theta = \frac{1}{2}\sec^2\theta \sin\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$

Example 7 Find $\int \frac{x\,dx}{a^2 + x^2}$.

Let $u = a^2 + x^2$. Then $du = 2x\,dx$,

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|a^2 + x^2| + C.$$

Since $a^2 + x^2$ is always positive

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln(a^2 + x^2) + C.$$

is equally correct.

Example 8 Find $\int \frac{dx}{\sqrt{x^2 - a^2}}$.

Assume $a > 0$. We make the trigonometric substitution $x = a\sec\theta$

Then $dx = a\tan\theta\sec\theta \, d\theta$, $\sqrt{x^2 - a^2} = a\tan\theta$.

$$\int \frac{dx}{\sqrt{x^2 - a^2}} d\theta = \int \frac{a\tan\theta\sec\theta}{a\tan\theta} d\theta = \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C'$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C' = \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + C'$$

Therefore $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

The formula $$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left|x + \sqrt{a^2 + x^2}\right| + C$$

can be derived in a similar way and is left as an exercise.

The integrals $\int \arctan x\,dx,\quad \int arc\sec x\,dx$

can now be evaluated using integration by parts,

$$\int u\,dv = uv - \int v\,du.$$

Example 9 Find $\int \arctan x\,dx.$

Let $u = \arctan x,\quad du = dx/(1 + x^2),\quad v = x,\quad dv = dx.$

Then $$\int \arctan x\,dx = \int u\,dv = uv - \int v\,du$$

$$= x\arctan x - \int \frac{x}{1 + x^2}\,dx.$$

From Example 7,

$$\int \frac{x\,dx}{1 + x^2} = \frac{1}{2}\ln(1 + x^2) + C.$$

Therefore $$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1 + x^2) + C.$$

$\int arc\cot x\,dx$ can be evaluated in a similar way.

Example 10 Find $\int arc\sec x\,dx$, when $x > 1$.

Let $u = arc\sec x,\quad du = \frac{1}{|x|\sqrt{x^2 - 1}}dx = \frac{1}{x\sqrt{x^2 - 1}}dx,\quad v = x,\quad dv = dx.$

Then $$\int arc\sec x\,dx = \int u\,dv = uv - \int v\,du = x\,arc\sec x - \int \frac{x}{x\sqrt{x^2 - 1}}dx$$

$$= x\,arc\sec x - \int \frac{1}{\sqrt{x^2 - 1}}dx.$$

From Example 8,

$$\int \frac{1}{\sqrt{x^2-1}}dx = \ln\left|x+\sqrt{x^2-1}\right| + C.$$

Therefore

$$\int arc\sec x\,dx = x\,arc\sec x - \ln\left|x+\sqrt{x^2-1}\right| + C.$$

8.8 INTEGRATION OF RATIONAL FUNCTIONS

Example 1 $\displaystyle\int \frac{x^3+4x^2-1}{x+2}dx.$

The first step is to divide the denominator into the number by long division.

$$\frac{x^3+4x^2-1}{x+2} = x^2+2x-4+\frac{7}{x+2}.$$

We now easily integrate each term in the sum.

$$\int \frac{x^3+4x^2-1}{x+2}dx = \int\left(x^2+2x-4+\frac{7}{x+2}\right)dx$$

$$= \frac{x^3}{3}+x^2-4x+7\ln|x+2|+C.$$

Example 2 $\displaystyle\int \frac{x^3+2x^2-20x-33}{x^2-3x-10}dx.$

Step 1 By long division, divide the denominator into the numerator. The result is

$$\frac{x^3+2x^2-20x-33}{x^2-3x-10} = x+5+\frac{5x+17}{x^2-3x-10}$$

Step 2 Break up the remainder $\dfrac{5x+17}{x^2-3x-10}$ into a sum,

$$\frac{5x+17}{x^2-3x-10} = \frac{-1}{x+2}+\frac{6}{x-5}.$$

One can readily check that Equation 1 is true,

$$\frac{-1}{x+2}+\frac{6}{x-5} = \frac{-(x-5)+6(x+2)}{(x+2)(x-5)} = \frac{5x+17}{x^2-3x-10}.$$

The terms $\dfrac{-1}{x+2}$ and $\dfrac{6}{x-5}$ are called partial fractions. Later on we shall explain how they were found. Notice that the denominators of the partial fractions are factors of the denominator of the rational function,

$$(x+2)(x-5) = x^2 - 3x - 10.$$

Step 3 We now have

$$\int \frac{x^3+2x^2-20x-33}{x^2-3x-10}dx = \int x\,dx + \int 5\,dx + \int -\frac{1}{x+2}dx + \int \frac{6}{x-5}dx$$

$$= \frac{x^2}{2} + 5x - \ln|x+2| + 6\ln|x-5| + C.$$

Example 3 $\int \frac{x^2}{x^3+3x^2+3x+1}dx$.

Step 1 This time the numerator already has smaller degree than the denominator, so no long division is needed.

Step 2 Break the rational function into a sum of *partial fractions*. T he denominator can be factored as

$$x^3+3x^2+3x+1 = (x+1)^3.$$

It turns out that

$$\frac{x^2}{(x+1)^3} = \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}$$

This can again be readily checked.

Step 3 $\int \frac{x^2}{(x+1)^3}dx = \int \frac{1}{x+1}dx + \int -\frac{2}{(x+1)^2}dx + \int \frac{1}{(x+1)^3}dx$

$$= \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$$

Example 4 $\int \frac{2x+3}{x^2+x+1}dx.$

Step 1 No long division is needed.

Step 2 The denominator x^2+x+1 cannot be factored, i.e., it is irreducible. In this case no sum of partial fractions is needed.

Step 3 To integrate $\int \frac{2x+3}{x^2+x+1}dx$

We use the method of *completing the square*. We have

$$x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4}.$$

Let $u = x + \frac{1}{2}$. Then $du = dx$ and

$$\int \frac{2x+3}{x^2+x+1}dx = \int \frac{2(u-\frac{1}{2})+3}{u^2+\frac{3}{4}}du = \int \frac{2u+2}{u^2+\frac{3}{4}}du$$

$$= \int \frac{2u}{u^2+\frac{3}{4}}du + \int \frac{2}{u^2+\frac{3}{4}}du$$

$$= \int \frac{d(u^2+\frac{3}{4})}{u^2+\frac{3}{4}} + 2\int \frac{1}{u^2+(\sqrt{3}/2)^2}du$$

$$= \ln\left|u^2+\frac{3}{4}\right| + \frac{4}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}u\right)+C$$

$$= \ln\left|x^2+x+1\right| + \frac{4}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)+C.$$

$$u = \left(\frac{\sqrt{3}}{2}\right)\tan\theta, \qquad \sqrt{u^2+\left(\sqrt{\frac{3}{2}}\right)^2} = \left(\frac{\sqrt{3}}{2}\right)\sec\theta$$

8.9 METHODS OF INTEGRATION

Example 1 $\displaystyle\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$

By multiplying the numerator and denominator by $\sqrt{x+1}+\sqrt{x}$ (i.e., rationalizing the denominator), we get the sum

$$\int \frac{dx}{\sqrt{x+1}-\sqrt{x}} = \int \frac{\sqrt{x+1}+\sqrt{x}}{(x+1)-x}dx = \int (\sqrt{x+1}+\sqrt{x})\,dx$$

$$= \int \sqrt{x+1}\,dx + \int \sqrt{x}\,dx.$$

Example 2 $\int \tan^3 x \sec^2 x\,dx.$ Using the identity $\sec^2 x = 1+\tan^2 x,$ we obtain a sum of integrals of powers of $\tan x$:

$$\int \tan^3 x\sec^2 x\,dx = \int \tan^3 x(1+\tan^2 x)\,dx = \int \tan^3 x\;dx + \int \tan^5 x\;dx.$$

Example 3 $\ln\left(\frac{x^2}{x+1}\right)dx$. Using the rules of logarithms we have

$$\int \ln\left(\frac{x^2}{x+1}\right)dx = \int (2\ln x - \ln(x+1))dx = 2\int \ln x\,dx - \int \ln(x+1)\,dx.$$

Example 4 $\int \sin(x+a)\sin(x-a)\,dx$. Using the addition formulas,

$$\sin(x+a) = \sin x\cos a + \cos x\sin a,$$
$$\sin(x-a) = \sin x\cos a - \cos x\sin a,$$

we have

$$\int \sin(x+a)\sin(x-a)$$

$$= \int (\sin x\cos a + \cos x\sin a)(\sin x\cos a - \cos x\sin a)dx$$

$$= \int (\sin^2 x\cos^2 a - \cos^2 x\sin^2 a)\,dx$$

$$= \cos^2 a\int \sin^2 x\,dx - \sin^2 a\int \cos^2 x\,dx.$$

The method of partial fractions also makes use of the Sum Rule.

Example 5 $\int \frac{x}{(x-a)(x-b)}dx,\quad a \neq 0,\quad b \neq 0.$ We have

$$\frac{x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b},$$

$$A = \frac{a}{a-b},\qquad B = \frac{b}{b-a},$$

$$\int \frac{x}{(x-a)(x-b)}dx = \frac{a}{a-b}\int \frac{dx}{x-a} + \frac{b}{b-a}\int \frac{dx}{x-b}.$$

Example 6 $\int \sqrt{2x+1}\,dx$. Let $u = 2x+1$, $du = 2dx$. Then

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\cdot\frac{1}{2}du.$$

This can be integrated using the Constant and Power Rules,

$$\int \sqrt{u}\cdot\frac{1}{2}du=\frac{\frac{1}{2}u^{3/2}}{\frac{3}{2}}=\frac{1}{3}u^{3/2}=\frac{1}{3}(2x+1)^{3/2}.$$

Example 7 $\int \frac{1}{\sqrt{x}+1}dx$. Let $u=\sqrt{x}$. Then $du=\frac{1}{2\sqrt{x}}dx,\quad dx=2u\,du$. We get the rational integral

$$\int \frac{1}{\sqrt{x}+1}dx=\int \frac{2u}{u+1}du.$$

Example 8 $\int \sin(3x^2-1)x\,dx$. Let $u=3x^2-1,\quad du=6x\,dx$. Then

$$\int \sin(3x^2-1)x\,dx=\int (\sin u)\frac{1}{6}du.$$

Example 9 $\int \frac{1}{2\sin x+\cos x}dx$. Putting $u=\tan\frac{x}{2}$, we obtain the rational integral

$$\int \frac{1}{\frac{4u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}du=\int \frac{2}{1+4u-u^2}du.$$

Example 10 $\int x^2\sqrt{x^2-6^2}\,dx$. We use the substitution $x=6\sec\theta$.

Then $\sqrt{x^2-6^2}=6\tan\theta$, $dx=6\tan\theta\sec\theta\,d\theta$, and the integral

becomes $\int 6^2\sec^2\theta\cdot 6\tan\theta\cdot 6\tan\theta\sec\theta\,d\theta$

$$=\int 6^4\tan^2\theta\sec^3\theta\,d\theta=6^4\int(\sec^2\theta-1)\sec^3\theta\,d\theta$$

$$=6^4\int\sec^5\theta\,d\theta-6^4\int\sec^3\theta\,d\theta.$$

Example 11 $\int x\ln x\,dx$. Try $u=\ln x,\quad dv=x\,dx$. Then

$$du=1/x\,dx,\qquad v=x^2/2,$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

Example 12 $\int (\ln x)^2 \, dx$. Put $u = (\ln x)^2$, $dv = x\,dx$. Then

$$du = \frac{2 \ln x}{x} dx, \qquad v = x,$$

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - 2\int \ln x \, dx.$$

Example 13 $\int \sin(\ln x) \, dx$. Let $u = \sin(\ln x)$, $dv = dx$. Then

$$du = \frac{\cos(\ln x)}{x} dx, \qquad v = x.$$

Integrating by parts,

$$\int \sin(\ln x) \, dx = x \sin(\ln x) - \int \cos(\ln x) \, dx.$$

Integrating by parts again,

$$\int \cos(\ln x) \, dx = x \cos(\ln x) + \int \sin(\ln x) \, dx.$$

Then

$$\int \sin(\ln x) \, dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \, dx,$$

$$\int \sin(\ln x) \, dx = \frac{1}{2} x \sin(\ln x) - \frac{1}{2} x \cos(\ln x) + C.$$