Maxima 在微積分上之應用

三角函數

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7.1 Trigonometry

7.2 Derivatives of Trigonometric Functions

Example 1.

Find the derivative of $y = \tan^2(3x)$.

Solution： $dy = 2\tan 3x d(\tan 3x) = 2\tan 3x \sec^2 3x d(3x)$

$= 6\tan 3x \sec^2 3x dx$.

Example 2.

Evaluate $\lim_{t\to\pi/2} \frac{\cos t}{t-\pi/2}$.

Solution：(%i1) limit(cos(t)/(t-%pi/2),t,%pi/2); 極限指令：limit(方程式，極限變數，範圍) //方程式爲 $\frac{\cos t}{t-\pi/2}$ ，極限變數爲 t，範圍爲 t 趨近於 $\pi/2$

(%o1) -1

This is a limit of the form 0/0 because

$$\lim_{t\to\pi/2} \cos t = 0, \quad \lim_{t\to\pi/2} (t-\frac{\pi}{2}) = 0.$$

By l'Hospital's Rule (Section 5.2)，

$$\lim_{t\to\pi/2} \frac{\cos t}{t-\pi/2} = \lim_{t\to\pi/2} \frac{-\sin t}{1} = -\sin(\frac{\pi}{2}) = -1.$$

Example 3.

A particle travels around a vertical circle of radius r_0 with constant angular velocity $\omega = d\theta/dt$, beginning with $\theta = 0$ at time $t = 0$. If the sun is directly overhead, find the position, velocity, and acceleration of the shadow.

Let us center the circle at the origin in the (x, y) plane (Figure 7.2.5). Then

$x = r_0 \cos\theta, \quad y = r_0 \sin\theta$.

At time t, θ has the value $\theta = \omega t$. So the motion of the particle is given by the parametric equations

$x = r_0 \cos(\omega t), \quad y = r_0 \sin(\omega t)$.

The shadow is directly below the particle, and its position is given by the x-component

$x = r_0 \cos(\omega t)$.

The velocity and acceleration of the shadow are

$$v = \frac{dx}{dt} = -r_0\omega\sin(\omega t),$$

$$a = \frac{dv}{dt} = -r_0\omega^2\cos(\omega t).$$

Example 4.

A light beam on a 100 ft tower rotates in a vertical circle at the rate of one revolution per second. Find the speed of the spot of light moving along the ground at a point 1000 ft from the base of the tower.

We start by drawing the picture in Figure 7.2.6.

Solution：

Assume the rotation is counterclockwise. Let t be time and let x and θ be as in the figure. Then

$$\frac{d\theta}{dt} = 2\pi \quad \text{radians/sec}, \quad x = 100\tan\theta \text{ ft}.$$

We wish to find dx/dt when $x = 1000$.

$$\frac{dx}{dt} = 100\sec^2\theta\frac{d\theta}{dt} = 200\pi\sec^2\theta.$$

When $x = 1000$,

$\sec^2\theta = 1 + \tan^2\theta = 1 + (x/100)^2 = 1 + 10^2 = 101.$

Therefore $\frac{dx}{dt} = 20200\pi \sim 63000$ ft/sec.

Example 5.

Find $\int \sin^3 t\cos t dt$. Let $u = \sin t$, $du = \cos t dt$.

Solution：(%i1) integrate(sin(t)^3*cos(t),t); 積分指令：integrate(數式，變數，範圍) 函數爲 $\sin^3 t\cos t$ ，變數爲 t，因爲不爲定積分所以不用打範圍

(%o1) $\frac{\sin(t)^4}{4}$

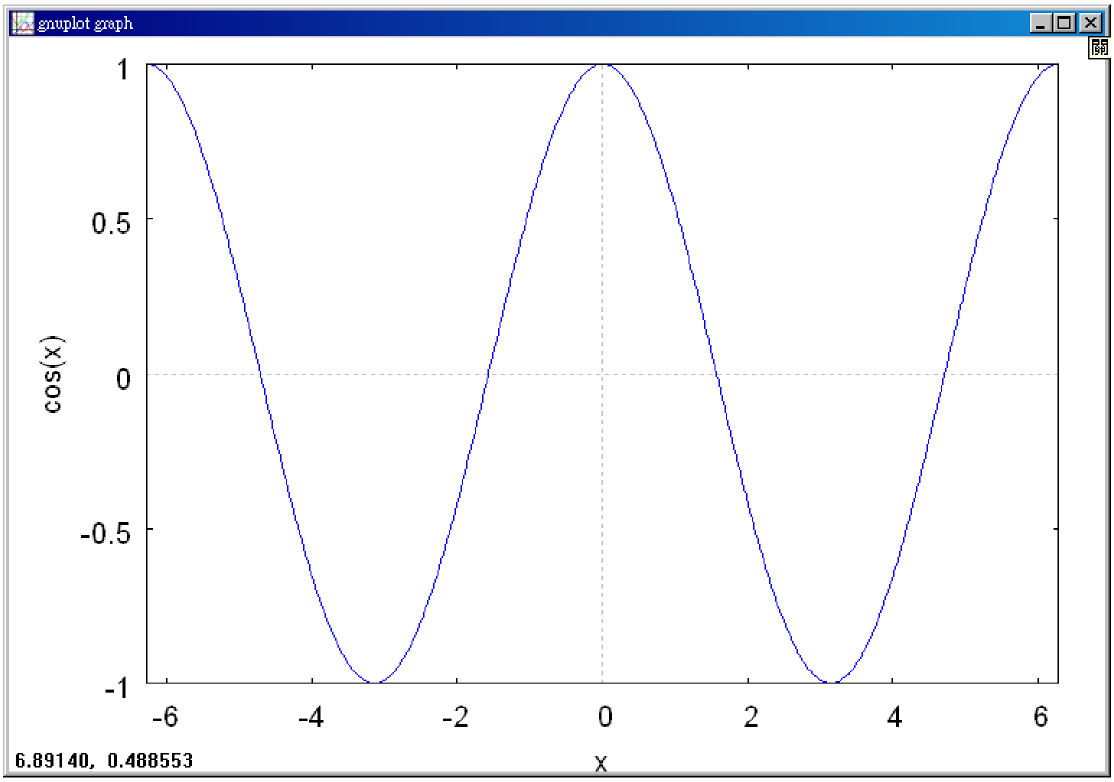
Then $\int \sin^3 t\cos t dt = \int u^3 du = \frac{u^4}{4} + C = \frac{\sin^4 t}{4} + C.$

Example 6.

Find the area under one arch of the curve $y = \cos x$.

Solution：(%i1) plot2d(cos(x),[x,-2*%pi,2*%pi]);   //畫出 $\cos x$ 的圖形，x 軸範圍為 $-2\pi \sim 2\pi$

(%o1)



(%i2) integrate(cos(x),x,-%pi/2,%pi/2);   積分指令：integrate(數式，變數，範圍)   函數為 $\sin^3 t\cos t$ ，變數為 t，定積分範圍為 $-\pi/2 \sim \pi/2$

(%o2) 2

From Figure 7.2.7 we see that one arch lies between the limits $x = -\pi/2$ and $x = \pi/2$, therefore the area is

$$\int_{-\pi/2}^{\pi/2} \cos t dt = \sin t\Big|_{-\pi/2}^{\pi/2} = 1 - (-1) = 2 .$$

Example 7.

Evaluate $\int \sec^4 x dx$.

Solution：(%i1) integrate(sec(x)^4,x);   積分指令：integrate(數式，變數，範圍)   函數為 $\sec^4 x$ ，變數為 x，因為不為定積分所以不用打範圍

(%o1) $\dfrac{\tan(x)^3}{3}+\tan(x)$

Using the identity $\sec^2 x = 1+\tan^2 x$, we have

$$\int \sec^4 x dx = \int (1+\tan^2 x)\sec^2 x dx$$

$$= \int (1+\tan^2 x) d(\tan x) = \tan x + \frac{\tan^3 x}{3} + C.$$

Example 8.

Find $\int \sqrt{1-\cos x} dx$.

Solution：(%i1) integrate(sqrt(1-cos(x)),x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\sqrt{1-\cos x}$ ，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $(-\sqrt{2}\cos(x)-\sqrt{2})\sin\left(\dfrac{x+\pi}{2}\right)+\sqrt{2}\sin(x)\cos\left(\dfrac{x+\pi}{2}\right)$

Using the identity $\sin^2 x + \cos^2 x = 1$, we have

$$\sqrt{1-\cos x} = \frac{\sqrt{1-\cos x}\sqrt{1+\cos x}}{\sqrt{1+\cos x}} = \frac{\sqrt{1-\cos^2 x}}{\sqrt{1+\cos^2 x}}$$

$$= \frac{\sqrt{\sin^2 x}}{\sqrt{1+\cos x}} = \frac{|\sin x|}{\sqrt{1+\cos x}}.$$

Case 1 In an interval where $\sin x \geq 0$,

$$\int \sqrt{10-\cos x} dx = \int \frac{\sin x}{\sqrt{1+\cos x}} dx = \int -\frac{1}{\sqrt{1+\cos x}} d(1+\cos x)$$

$$= -2\sqrt{1+\cos x} + C.$$

Case 2 In an interval where $\sin x \leq 0$,

$$\int \sqrt{1-\cos x} dx = 2\sqrt{1+\cos x} + C.$$

7.3 Inverse Trigonometric Functions

Example 1.

Find $\arccos(\sqrt{2}/2)$.

Solution：(%i1) plot2d(acos(y),[y,-2,2]);　//畫出 arccos 圖形，y 軸範圍為-2~2

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plot2d: expression evaluates to non-numeric value somewhere in plot
  (%o1)
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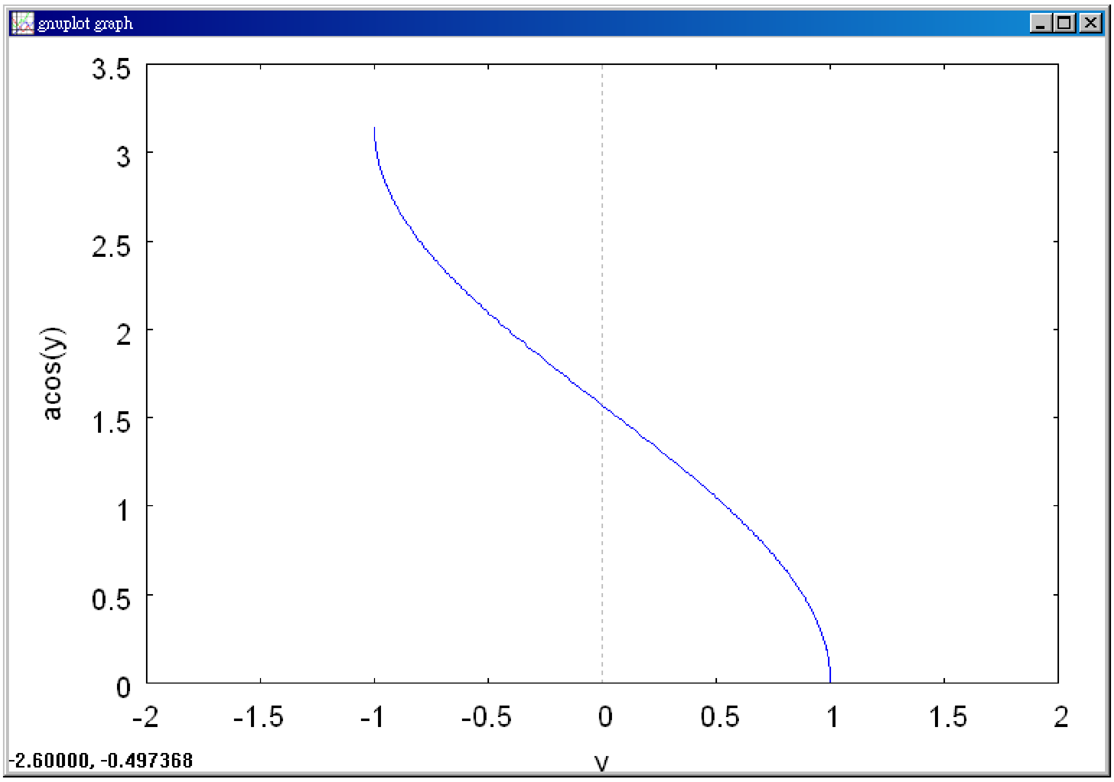

(%i2) acos(sqrt(2)/2);　arccos 指令：acos(數值)　//此題為求 arccos($\sqrt{2}/2$)的值

(%o2) $\frac{\pi}{4}$

From Table 7.1.1, $\cos(\pi/4) = \sqrt{2}/2$. Since $0 \le \pi/4 \le \pi$,

$\arccos(\sqrt{2}/2) = \pi/4$.

Example 2.

Find $\arcsin(-1)$.

Solution：(%i1) plot2d(asin(y),[y,-2,2]);　//畫出 arcsin 圖形，y 軸範圍為-2~2

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plot2d: expression evaluates to non-numeric value somewhere in plott
 (%o1)
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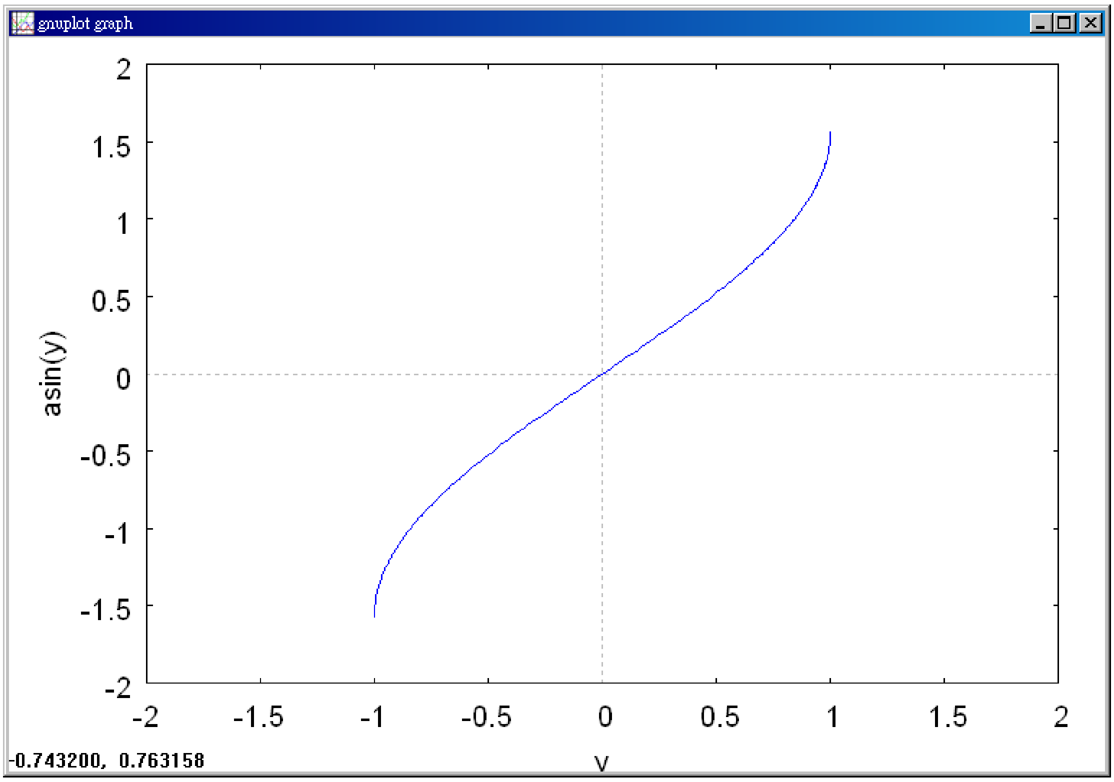


(%i2) asin(-1); arcsin 指令：asin(數值) //此題爲求 arcsin(1)的值

(%o2) $-\frac{\pi}{2}$

From Table 7.1.1, $\sin(3\pi/2) = -1$. But $3\pi/2$ is not in the interval $[-\pi/2, \pi/2]$. Using $\sin(\theta + 2n\pi) = \sin\theta$, we have

$$\sin(-\pi/2) = \sin(3\pi/2) = -1,$$

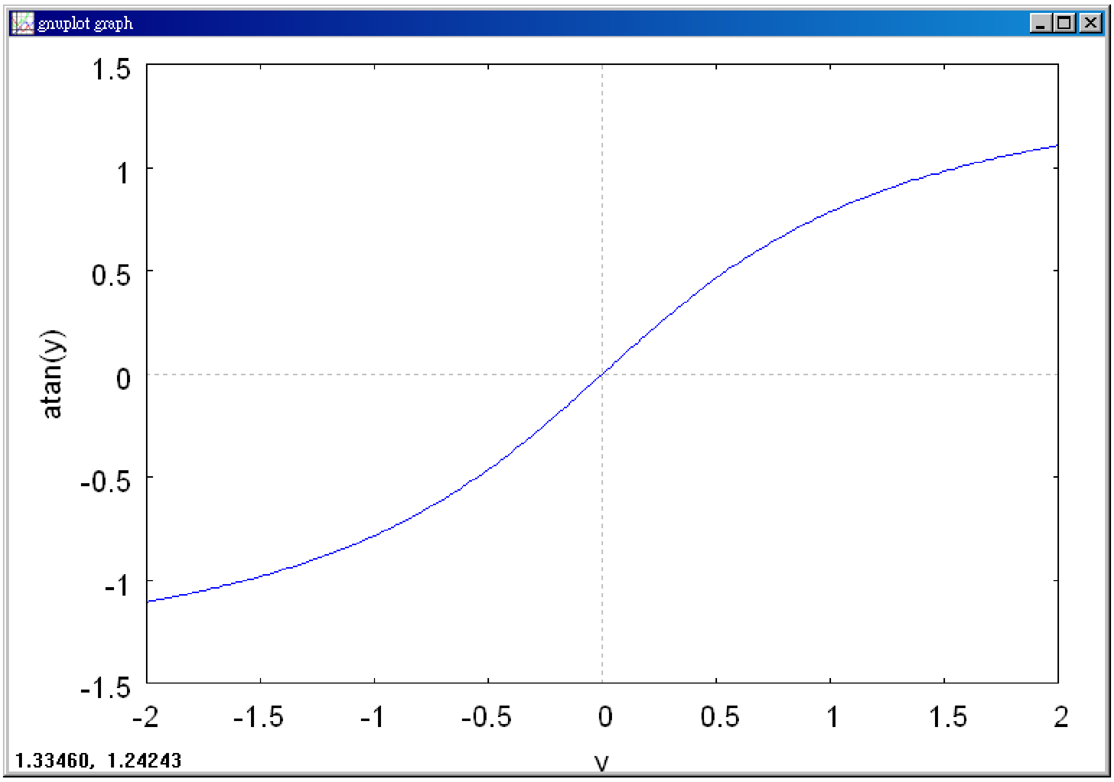
So $\arcsin(-1) = -\pi/2$.

Example 3.

Find $\arctan(-\sqrt{3})$.

Solution：(%i1) plot2d(atan(y),[y,-2,2]);　//畫出 arctan 圖形，y 軸範圍爲-2~2

(%o1)



(%i2) atan(-sqrt(3));　arctan 指令：atan(數值)　//此題爲求 arctan($-\sqrt{3}$)的值

(%o2) $-\frac{\pi}{3}$

We must find a θ in the interval $[-\pi/2, \pi/2]$ such that $\tan\theta = -\sqrt{3}$. From Table 7.1.1, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/3) = 1/2$.

Then $\sin(-\pi/3) = -\sqrt{3}/2$, $\cos(-\pi/3) = 1/2$. So

$$\tan(-\pi/3) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3},$$

$\arctan(-\sqrt{3}) = -\pi/3$.

Example 4.

Find $\cos(\arctan y)$.

Solution：(%i1) cos(atan(y));

(%o1) $\frac{1}{\sqrt{y^2+1}}$

Let $\theta = \arctan y$. Thus $\tan\theta = y$. Using

$\sin^2\theta + \cos^2\theta = 1$

$\frac{\sin\theta}{\cos\theta} = y$,

we solve for $\cos\theta$.

$\sin\theta = y\cos\theta, \quad (y\cos\theta)^2 + \cos^2\theta = 1$,

$\cos^2\theta(y^2+1) = 1, \quad \cos^2\theta = \frac{1}{y^2+1}$.

Thus $\cos\theta = \pm\frac{1}{\sqrt{y^2+1}}$.

By definition of $\arctan y$, we know that $-\pi/2 \le \theta \le \pi/2$. In this interval,

$\cos\theta \ge 0$. Therefore $\cos\theta = \frac{1}{\sqrt{y^2+1}}$.

Example 5.

Show that $\arcsin y + \arccos y = \pi/2$ (Figure 7.3.6). Let $\theta = \arcsin y$.

We have $y = \sin\theta = \cos(\pi/2 - \theta)$. Also, when $-\pi/2 \le \theta \le \pi/2$, we have

$\pi/2 \ge -\theta \ge -\pi/2, \quad \pi \ge \pi/2 - \theta \ge 0$.

Thus $\pi/2 - \theta = \arccos y$,

$\arcsin y + \arccos y = \theta + (\pi/2 - \theta) = \pi/2$.

Example 6.

(a) Find the area of the region under the curve $y = \dfrac{1}{1+x^2}$ for $-1 \le x \le 1$.

(b) Find the area of the region under the same curve for $-\infty < x < \infty$. The regions are shown in Figure 7.3.7.

Solution：

(a) (%i1) integrate(1/(1+x^2),x,-1,1); 積分指令：integrate(數式，變數，範圍) 函數爲$\dfrac{1}{1+x^2}$，變數爲 x，定積分範圍爲-1~1

(%o1) $\dfrac{\pi}{2}$

$$A = \int_{-1}^{1} \frac{1}{1+x^2} dx = \arctan x \Big|_{-1}^{1} = \frac{\pi}{4} - (-\frac{\pi}{4}) = \frac{\pi}{2}.$$

(b) (%i1) integrate(1/(1+x^2),x,minf,inf); 積分指令：integrate(數式，變數，範圍) 函數爲$\dfrac{1}{1+x^2}$，變數爲 x，定積分範圍爲$-\infty \sim \infty$

(%o1) π

$$A = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2} dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2} dx$$

$$= \lim_{a\to-\infty}(\arctan 0 - \arctan a) + \lim_{b\to\infty}(\arctan b - \arctan 0)$$

$$= -\lim_{a\to-\infty}\arctan a + \lim_{b\to\infty}\arctan b\,.$$

From the graph of $\arctan x$ we see that the first limit is $-\pi/2$ and the second limit is $\pi/2$, so

$$A = -(-\frac{\pi}{2}) + \frac{\pi}{2} = \pi\,.$$

Thus the region under $y = 1/(1+x^2)$ has exactly the same area as the unit circle, and half of this area is between $x=-1$ and $x=1$.

Example 7.

Find $\int_{-2}^{-\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}dx$.

Solution：(%i1) integrate(1/(x*sqrt(x^2-1)),x,-2,-sqrt(2)); 積分指令：integrate(數式，變數，範圍) 函數爲 $\frac{1}{x\sqrt{x^2-1}}$，變數爲 x，定積分範圍爲 $-2 \sim -\sqrt{2}$

(%o1) $-\frac{\pi}{12}$

The region is shown in Figure 7.3.8. Since x is negative, $x = -|x|$. Thus

$$\int_{-2}^{-\sqrt{2}} \frac{1}{x\sqrt{x^2-1}}dx = \int_{-2}^{-\sqrt{2}} \frac{1}{|x|\sqrt{x^2-1}}dx$$

$$= -arc\sec x\Big|_{-2}^{-\sqrt{2}} = -(arc\sec(-\sqrt{2}) - arc\sec(-2))$$

$$= -(\frac{3\pi}{4} - \frac{2\pi}{3}) = -\frac{\pi}{12}\,.$$

7.4 Integration by Parts

Example 1.

Evaluate $\int x \sin dx$. Our plan is to break $x \sin xdx$ into a product of the form udv,

evaluate the integrals $\int dv$ and $\int vdu$, and then use integration by parts to get $\int udv$.

There are several choices we might make for u and dv, and not all of them lead to a solution of the problem. Some guesswork is required.

Solution：(%i1) integrate(x*sin(x),x); 積分指令：integrate(數式，變數，範圍) 函數爲 $x \sin x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) sin(x)−x cos(x)

First try： $u = \sin x, dv = xdx$. $\int dv = \int xdx = \frac{1}{2}x^2 + C$. Take $v = \frac{1}{2}x^2$.

Next we find du and try to evaluate $\int vdu$.

$du = \cos xdx, \quad \int vdu = \int \frac{1}{2}x^2 \cos xdx$.

This integral looks harder than the one we started with, so we shall start over with another choice of u and dv.

Second try： $u = x, dv = \sin xdx.$

$\int dv = \int \sin xdx = -\cos x + C$.

We take $v = -\cos x$. This time we find du and easily evaluate $\int vdu$.

$du = dx, \quad \int vdu = \int -\cos xdx = -\sin x + C_1$.

Finally we use the rule

$\int udv = uv - \int vdu$,

$\int x\sin xdx = x(-\cos x) - (-\sin x + C_1)$,

or $\int x\sin xdx = -x\cos x + \sin x + C$.

Example 2.

Evaluate $\int \arcsin xdx$. A choice of u and dv which works is

$u = \arcsin x$, $dv = dx$.

Solution：(%i1) integrate(asin(x),x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\arcsin x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $x\,\mathrm{asin}(x)+\sqrt{1-x^2}$

We may take $v = x$. Then

$$du = \frac{dx}{\sqrt{1-x^2}},$$

$$\int vdu = \int \frac{xdx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C_1.$$

Finally, $\int \arcsin xdx = x\arcsin x - (-\sqrt{1-x^2} + C_1)$,

$$\int \arcsin xdx = x\arcsin x + \sqrt{1-x^2} + \mathrm{C}.$$

Example 3.

Evaluate $\int x^2 \sin x dx$. This requires two integrations by parts.

Solution：(%i1) integrate(x^2*sin(x),x); 積分指令：integrate(數式，變數，範圍)
函數爲 $x^2 \sin x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $2\, x \sin(x) + (2 - x^2) \cos(x)$

Step 1： $u = x^2, \quad dv = \sin x dx$,

$$du = 2x dx, \quad \int dv = \int \sin x dx = -\cos x + C .$$

We take $v = -\cos x$.

$$\int x^2 \sin x dx = uv - \int v du = -x^2 \cos x + \int 2x \cos x dx .$$

Step 2： Evaluate $\int 2x \cos x dx$.

$u_1 = 2x, \quad dv_1 = \cos x dx$,

$$du_1 = 2dx, \quad \int dv_1 = \int \cos x dx = \sin x + C .$$

We take $v_1 = \sin x$.

$$\int 2x \cos x dx = u_1 v_1 - \int v_1 du_1$$

$$= 2x \sin x - \int 2 \sin x dx$$

$$= 2x \sin x + 2 \cos x + C .$$

Combining the two steps,

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C .$$

Example 4.

Evaluate $\int \sin^2\theta d\theta$.

Solution：(%i1) integrate(sin(x)^2,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\sin^2 x$ ，變數爲 x，因爲不爲定積分所以不用打範圍，由於 θ 在 Maxima 中部好表示，於是我們用 x 代替 θ 這個變數

(%o1) $\dfrac{x-\dfrac{\sin(2x)}{2}}{2}$

Let $u=\sin\theta,\quad dv=\sin\theta d\theta$.

Then $du=\cos\theta d\theta,\quad v=-\cos\theta$.

$$\int \sin^2\theta d\theta = -\sin\theta\cos\theta - \int -\cos^2\theta d\theta$$

$$= -\sin\theta\cos\theta + \int \cos^2\theta d\theta$$

$$= -\sin\theta\cos\theta + \int (1-\sin^2\theta) d\theta$$

$$= -\sin\theta\cos\theta + \theta - \int \sin^2\theta d\theta.$$

We solve this equation for $\int \sin^2\theta d\theta$,

$$\int \sin^2\theta d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C.$$

Example 5.

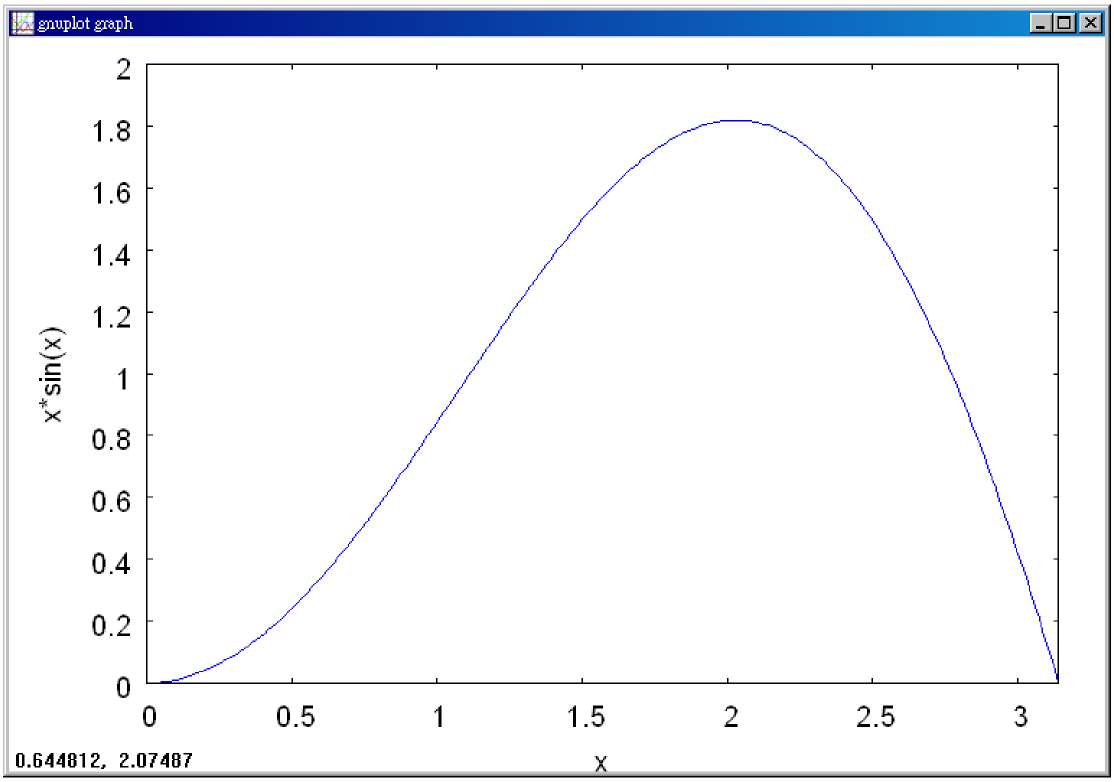
Evaluate $\int_0^{\pi} x\sin x dx$ (Figure 7.4.2).

Solution：(%i1) integrate(x*sin(x),x,0,%pi); 積分指令：integrate(數式，變數，範圍) 函數爲 $x\sin x$ ，變數爲 x，定積分範圍爲 0~π

(%o1) π

(%i2) plot2d(x*sin(x),[x,0,%pi]); //畫出 $x\sin x$ 的圖形，x 軸範圍爲 0~π

(%o2)



Take $u = x$, $dv = \sin x dx$ as in Example 1. Then $v = -\cos x$ and

$$\int_0^{\pi} x\sin x dx = -x\cos x\Big|_0^{\pi} - \int_0^{\pi} -\cos x dx$$

$$= -x\cos x\Big|_0^{\pi} + \sin x\Big|_0^{\pi}$$

$$= (-\pi(-1) + 0\cdot 1) + (0-0) = \pi .$$

7.5 Integrals of Powers of Trigonometric Functions

Example 1.

$$\int \tan^2 xdx = \frac{\tan x}{1} - \int \tan^0 xdx = \tan x - x + C.$$

(%i1) integrate(tan(x)^2,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\tan^2 x$，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) tan(x)−atan(tan(x))

Example 2.

$$\int \tan^4 xdx = \frac{\tan^3 x}{3} - \int \tan^2 xdx = \frac{\tan^3 x}{3} - \tan x + x + C.$$

(%i1) integrate(tan(x)^4,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\tan^4 x$，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $\text{atan}(\tan(x)) + \frac{\tan(x)^3 - 3\tan(x)}{3}$

Example 3.

$$\int \tan^3 xdx = \frac{\tan^2 x}{2} - \int \tan xdx.$$

(%i1) integrate(tan(x)^3,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\tan^3 x$，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $\frac{\log(\sin(x)^2 - 1)}{2} - \frac{1}{2\sin(x)^2 - 2}$

Example 4.

$$\int \sin^2 xdx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}\int dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C.$$

$$\int \cos^2 xdx = \frac{1}{2}\cos x\sin x + \frac{1}{2}\int dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}x + C.$$

(%i1) integrate(sin(x)^2,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\sin^2 x$，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $\dfrac{x-\dfrac{\sin(2x)}{2}}{2}$

(%i2) integrate(cos(x)^2,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\cos^2 x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

(%o2) $\dfrac{\dfrac{\sin(2x)}{2}+x}{2}$

Example 5.

$$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x dx$$

$$= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C .$$

(%i1) integrate(cos(x)^3,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\cos^3 x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $\sin(x)-\dfrac{\sin(x)^3}{3}$

Example 6.

$$\int \sec^3 x dx = \frac{1}{2}\sec^2 x \sin x + \frac{1}{2}\int \sec x dx .$$

(%i1) integrate(sec(x)^3,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\sec^3 x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $\dfrac{\log(\sin(x)+1)}{4}-\dfrac{\log(\sin(x)-1)}{4}-\dfrac{\sin(x)}{2\sin(x)^2-2}$

Example 7.

$$\int \sec^4 x dx = \frac{1}{3}\sec^3 x \sin x + \frac{2}{3}\int \sec^2 x dx$$

$$=\frac{1}{3}\sec^3 x\sin x+\frac{2}{3}\tan x+C.$$

(%i1) integrate(sec(x)^4,x); 積分指令：integrate(數式，變數，範圍) 函數為 $\sec^4 x$，變數為 x，因為不為定積分所以不用打範圍

(%o1) $\frac{\tan(x)^3}{3}+\tan(x)$

Example 8.

$\int \sin^4 x\cos^3 xdx$.

Solution：(%i1) integrate((sin(x)^4)*(cos(x)^3),x); 積分指令：integrate(數式，變數，範圍) 函數為 $\sin^4 x\cos^3 x$，變數為 x，因為不為定積分所以不用打範圍

(%o1) $-\frac{5\sin(x)^7-7\sin(x)^5}{35}$

Let $u=\sin x,\quad du=\cos xdx$.

$$\int \sin^4 x\cos^3 xdx=\int u^4(1-u^2)du$$

$$=\frac{1}{5}u^5-\frac{1}{7}u^7+C$$

$$=\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+C.$$

Example 9.

$\int \sqrt{\cos x}\sin^3 xdx$.

Solution：(%i1) integrate(sqrt(cos(x))*(sin(x)^3),x); 積分指令：integrate(數式，變數，範圍) 函數為 $\sqrt{\cos x}\sin^3 x$，變數為 x，因為不為定積分所以不用打範圍

(%o1) $\frac{6\cos(x)^{7/2}-14\cos(x)^{3/2}}{21}$

Let $u=\cos x,\quad du=-\sin xdx$.

$$\int \sqrt{\cos x}\sin^3 xdx=\int \sqrt{u}(1-u^2)(-1)du$$

$$= -u^{1/2} + u^{5/2} du = -\frac{2}{3}u^{3/2} + \frac{2}{7}u^{7/2} + C$$

$$= -\frac{2}{3}(\cos x)^{3/2} + \frac{2}{7}(\cos x)^{7/2} + C.$$

Example 10.

$\int \sin^5 x dx$.

Solution：(%i1) integrate(sin(x)^5,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\sin^5 x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

$$\text{(\%o1)} \quad -\frac{\cos(x)^5}{5} + \frac{2\cos(x)^3}{3} - \cos(x)$$

Let $u = \cos x, \quad du = -\sin x dx$.

$$\int \sin^5 x dx = \int (1-u^2)^2 (-1) du$$

$$= -\int (1-2u^2+u^4) du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.$$

Example 11.

$$\int \sin^4 \cos^4 x dx = \int \sin^4 x(1-\sin^2 x)^2 dx$$

$$= \int \sin^4 x - 2\sin^6 x + \sin^8 x dx.$$

(%i1) integrate((sin(x)^4)*(cos(x)^4),x); 積分指令：integrate(數式，變數，範圍)
函數爲 $\sin^4 x\cos^4 x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

$$(\%o1)\quad \frac{\frac{\frac{\sin(8x)}{2}+4x}{8}-\frac{\sin(4x)}{2}+x}{64}$$

Example 12.

When m is even use $\tan^2 x=\sec^2 x-1$.

$$\int \tan^4 x\sec x dx=\int(\sec^2 x-1)^2\sec x dx$$

$$=\int \sec^5 x-2\sec^3 x+\sec x dx.$$

(%i1) integrate((tan(x)^4)*(sec(x)),x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\tan^4 x\sec x$ ，變數爲 x，因爲不爲定積分所以不用打範圍

$$(\%o1)\quad \frac{3\log(\sin(x)+1)}{16}-\frac{3\log(\sin(x)-1)}{16}+\frac{5\sin(x)^3-3\sin(x)}{8\sin(x)^4-16\sin(x)^2+8}$$

Example 13.

When m is odd use the new variable $u=\sec x$ or $u=-\csc x$.

$$\int \cot^3 x\csc^3 x dx=\int \cot^2 x\csc^2 x(\cot x\csc x dx)$$

$$=\int(u^2-1)u^2 du=\frac{u^5}{5}-\frac{u^3}{3}+C$$

$$=-\frac{\csc^5 x}{5}+\frac{\csc^3 x}{3}+C.$$

7.6 Trigonometric Substitutions

Example 1.

Find $\int (a^2+x^2)^{-3/2} dx$.

Solution：(%i1) integrate((a^2+x^2)^(-3/2),x); 積分指令：integrate(數式，變數，範圍) 函數爲 $(a^2+x^2)^{-3/2}$ ，變數爲 x，因爲不爲定積分所以不用打範圍

(%o1) $\dfrac{x}{a^2\sqrt{x^2+a^2}}$

Let $\theta = \arctan(x/a)$. Then from Figure 7.6.2,

$$x = a\tan\theta,\quad dx = a\sec^2\theta d\theta,\quad \sqrt{a^2+x^2} = a\sec\theta.$$

So

$$\int (a^2+x^2)^{-3/2} dx = \int (a\sec\theta)^{-3} a\sec^2\theta d\theta$$

$$= \frac{1}{a^2}\int (\sec\theta)^{-1} d\theta = \frac{1}{a^2}\int \cos\theta d\theta$$

$$= \frac{1}{a^2}\sin\theta + C = \frac{1}{a^2}\frac{\tan\theta}{\sec\theta} + C = \frac{x}{a^2\sqrt{a^2+x^2}} + C.$$

Example 2.

Find $\int \sqrt{x^2-a^2}\, dx$.

Solution：(%i1) integrate(sqrt(x^2-a^2),x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\sqrt{x^2-a^2}$ ，變數爲 x，因爲不爲定積分所以不用打範圍

$$(\%o1)\quad \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2\log\left(2\sqrt{x^2-a^2}+2x\right)}{2}$$

Let $\theta = arc\sec(x/a)$ (Figure 7.6.3), so

$$x = a\sec\theta,\quad dx = a\tan\theta\sec\theta d\theta,\quad \sqrt{x^2-a^2} = a\tan\theta.$$

So

$$\int\sqrt{x^2-a^2}dx = \int a\tan\theta a\tan\theta\sec\theta d\theta = a^2\int\tan^2\theta\sec\theta d\theta$$

$$= a^2\int(\sec^2\theta-1)\sec\theta d\theta$$

$$= a^2\int\sec^3\theta d\theta - a^2\int\sec\theta d\theta$$

$$= (\frac{1}{2}a^2\sec^2\theta\sin\theta + \frac{1}{2}a^2\int\sec\theta d\theta) - a^2\int\sec\theta d\theta$$

$$= \frac{1}{2}a^2\sec^2\theta\sin\theta - \frac{1}{2}a^2\int\sec\theta d\theta$$

$$= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\int\sec\theta d\theta$$

Example 3.

$$\int\frac{1}{x^2\sqrt{a^2-x^2}}dx.$$

Solution：(%i1) integrate(1/(x^2*(sqrt(a^2-x^2))),x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\frac{1}{x^2\sqrt{a^2-x^2}}$，變數爲 x，因爲不爲定積分所以不用打範圍，第二式是問 a 是否爲 0 或不爲零的數，在此我們令 a 爲不爲零的數

Is a zero or nonzero?nonzero;

(%o1) $-\frac{\sqrt{a^2-x^2}}{a^2 x}$

Let $\theta = \arcsin(x/a)$ (Figure 7.6.4). Then

$$x = a\sin\theta, \quad dx = a\cos\theta d\theta, \quad \sqrt{a^2-x^2} = a\cos\theta .$$

$$\int \frac{1}{x^2\sqrt{a^2-x^2}}dx = \int \frac{1}{a^2\sin^2\theta a\cos\theta}a\cos\theta d\theta = \int \frac{1}{a^2\sin^2\theta}d\theta$$

$$= \frac{1}{a^2}\int \sec^2\theta d\theta = -\frac{1}{a^2}\cot\theta + C$$

$$= -\frac{1}{a^2}\frac{\sqrt{a^2-x^2}}{x} + C .$$

Example 4.

$$\int \frac{\sqrt{x^2-a^2}}{x}dx .$$

Solution：(%i1) integrate((sqrt(x^2-a^2))/x,x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\frac{\sqrt{x^2-a^2}}{x}$，變數爲 x，因爲不爲定積分所以不用打範圍，第二式是問 a 是否爲 0 或不爲零的數，在此我們令 a 爲不爲零的數

Is a zero or nonzero?nonzero;

(%o1) $\frac{a^2 \operatorname{asin}\left(\frac{a^2}{|a|\,|x|}\right)}{|a|} + \sqrt{x^2-a^2}$

Put $\theta = arc\sec(x/a)$ (Figure 7.6.5). Then

$x = a\sec\theta, \quad dx = a\tan\theta\sec\theta d\theta, \quad \sqrt{x^2 - a^2} = a\tan\theta$.

$$\int \frac{\sqrt{x^2-a^2}}{x}dx = \int \frac{a\tan\theta}{a\sec\theta} a\tan\theta\sec\theta d\theta = a\int \tan^2\theta d\theta$$

$$= a\int \sec^2\theta d\theta - a\int d\theta = a\tan\theta - a\theta + C$$

$$= \sqrt{x^2-a^2} - a\,arc\sec(x/a) + C.$$

Example 5.

The basic integrals：

(a) $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$,

(b) $\int \frac{dx}{1+x^2}dx = \arctan x + C$,

(c) $\int \frac{dx}{x\sqrt{x^2-1}} = arc\sec x + C, \quad x > 1$

can be evaluated very easily by a trigonometric substitution.

Solution：

(a) (%i1) integrate(1/sqrt(1-x^2),x); 積分指令：integrate(數式，變數，範圍) 函數為 $\frac{1}{\sqrt{1-x^2}}$ ，變數為 x，因為不為定積分所以不用打範圍

```
(%o1) asin(x)
```

$$\int \frac{1}{\sqrt{1-x^2}}dx.$$

Let $\theta = \arcsin x$ (Figure 7.6.6). Then $x = \sin\theta, \quad dx = \cos\theta d\theta, \quad \sqrt{1-x^2} = \cos\theta$.

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{\cos\theta d\theta}{\cos\theta} = \int d\theta = \theta + C,$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C.$$

(b) (%i1) (%i1) integrate(1/(1+x^2),x); 積分指令:integrate(數式,變數,範圍) 函數爲 $\frac{1}{1+x^2}$ ，變數爲 x，因爲不爲定積分所以不用打範圍

```
(%o1) atan(x)
```

$$\int \frac{dx}{1+x^2}.$$

Let $\theta = \arctan x$ (Figure 7.6.7. Then $x = \tan\theta$, $dx = \sec^2\theta d\theta$, $\sqrt{1+x^2} = \sec\theta$.

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2\theta}{\sec^2\theta} d\theta = \int d\theta = \theta + C,$$

$$\int \frac{dx}{1+x^2} = \arctan x + C.$$

(c) (%i1) integrate(1/(x*sqrt(x^2-1)),x); 積分指令：integrate(數式，變數，範圍) 函數爲 $\frac{1}{x\sqrt{x^2-1}}$ ，變數爲 x，因爲不爲定積分所以不用打範圍

$$\text{(\%o1)} \quad -\mathrm{asin}\left(\frac{1}{|x|}\right)$$

$$\int \frac{dx}{x\sqrt{x^2-1}}, \quad x > 1.$$

Let $\theta = arc\sec x$ (Figure 7.6.8). Then

$x = \sec\theta$, $dx = \tan\theta\sec\theta d\theta$, $\sqrt{x^2-1} = \tan\theta$.

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\tan\theta\sec\theta}{\sec\theta\tan\theta} d\theta = \int d\theta = \theta + C.$$

$$\int \frac{dx}{x\sqrt{x^2}+1} = arc\sec x + C, \quad x > 1.$$

7.7 Polar Coordinates

Example 1.

Plot the following points in polar coordinates.

(2, $\pi/4$), (-1, $\pi/4$), (3, $3\pi/4$), (2, -π/4), (-4, -$\pi/4$).

The solution is shown in Figure 7.7.3.

Each point P has infinitely many different polar coordinate pairs. We see in Figure 7.7.4 that the point $P(3,\pi/2)$ has all the coordinates

$$\left.\begin{array}{l}(3,\pi/2+2n\pi),\\(-3,3\pi/2+2n\pi),\end{array}\right\} n \quad \text{an integer.}$$

Example 2.

The graph of the equation $r = a$ is the circle of radius a centered at the origin (Figure 7.7.6(a)). The graph of the equation $\theta = b$ is a straight line through the origin (Figure 7.7.6(b)).

Example 3.

The graph of the system of formulas

$$r = \theta, \quad 0 \le \theta$$

is the spiral of Archimedes formed by moving a pencil along the line OX while the line is rotating with the pencil moving at the same speed as the point X. The graph is shown in Figure 7.7.6(c).

Example 4.

The parabola $y = x^2$ has the polar equation

$$r\sin\theta = (r\cos\theta)^2, \quad \text{or} \quad r = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta.$$

Example 5.

The curve $y = 1/x$ has the polar equation

$$r\sin\theta = \frac{1}{r\cos\theta}, \quad \text{or} \quad r^2 = \sec\theta\csc\theta.$$

The graph is shown in Figure 7.7.8.

Example 6.

The graph of the equation

$$r = a\sin\theta$$

is the circle one of whose diameters is the line from the origin to a point a above the origin.

This can be seen from Figure 7.7.9, if we remember that a diameter and a point on the circle form a right triangle.

As θ increase, the point $(a\sin\theta, \theta)$ goes around this circle once for every π radians.

Example 7.

(a) The spiral $r = \theta$ has the parametric equations

$x = \theta\cos\theta, \quad y = \theta\sin\theta$.

(b) The circle $r = a\sin\theta$ has the parametric equations

$x = a\sin\theta\cos\theta, \quad y = a\sin^2\theta$.

7.8 Slopes and Curve Sketching in Polar Coordinates

Example 1.

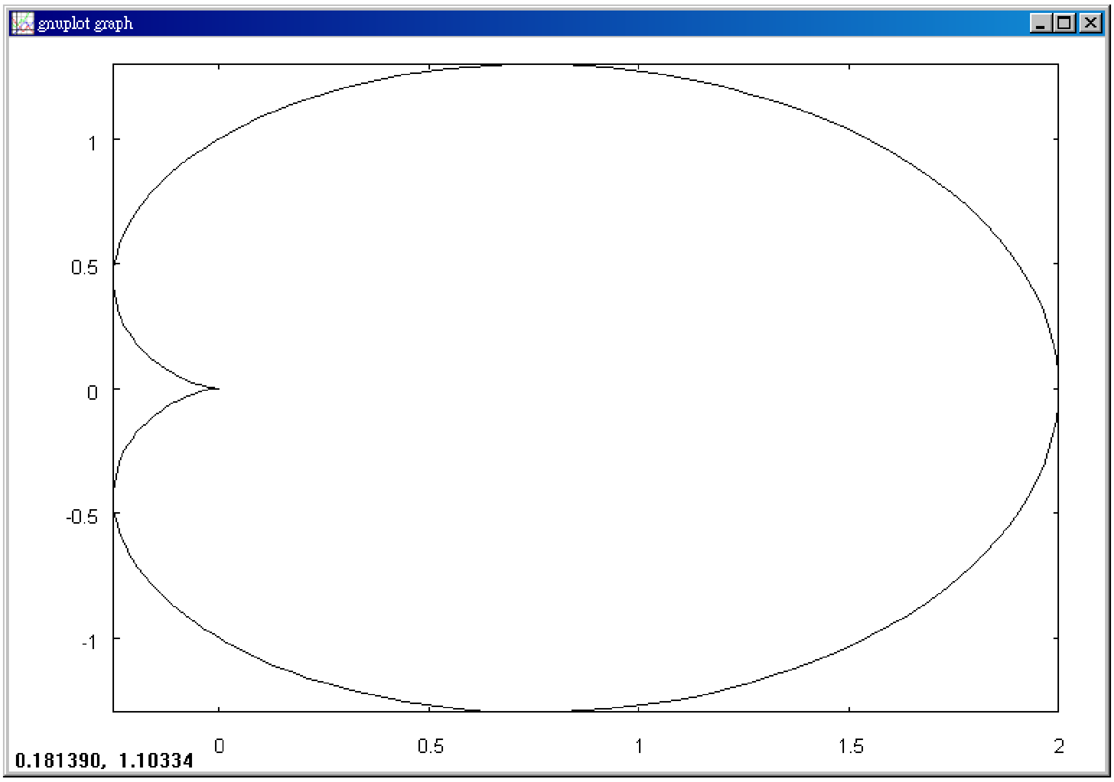
Sketch the curve $r = 1 + \cos\theta$.

Solution：(%i1) load(draw); 由於極坐標畫圖並沒有直接的指令，於是我們需要讀取模組 draw，此模組裡頭包含極坐標畫圖的指令 //讀取模組 draw

```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp
```

(%i2) draw2d(nticks=200,polar(1+cos(theta),theta,0,2*%pi)); 極坐標畫圖指令：draw(nticks=點數，polar(函數式，角度，最小的角度，最大的角度)) //在此題畫極坐標表示的函數圖形爲$1+\cos\theta$，角度爲θ，最小的角度是 0，最大的角度是 2π，用 200 個點描繪曲線

```
(%o2) [gr2d(polar)]
```



Step 1： $dr/d\theta = -\sin\theta$.

Step 2： $r = 0$ when $\theta = \pi$. $dr/d\theta = 0$ where $\theta = 0, \pi$.

Step 3： See Figure 7.8.6.

Step 4：

θ	$r = 1 + \cos\theta$	$dr/d\theta$	$\tan\psi$	Comments
0	2	0	—	max
$\pi/2$	1	-1	-1	$\lvert r \rvert$ decreasing
π	0	0	—	Min, cusp at 0
$3\pi/2$	1	1	1	$\lvert r \rvert$ increasing

Step 5： We draw the curve in Figure 7.8.7. The curve is called a cardioid because of its heart shape.

Example 2.

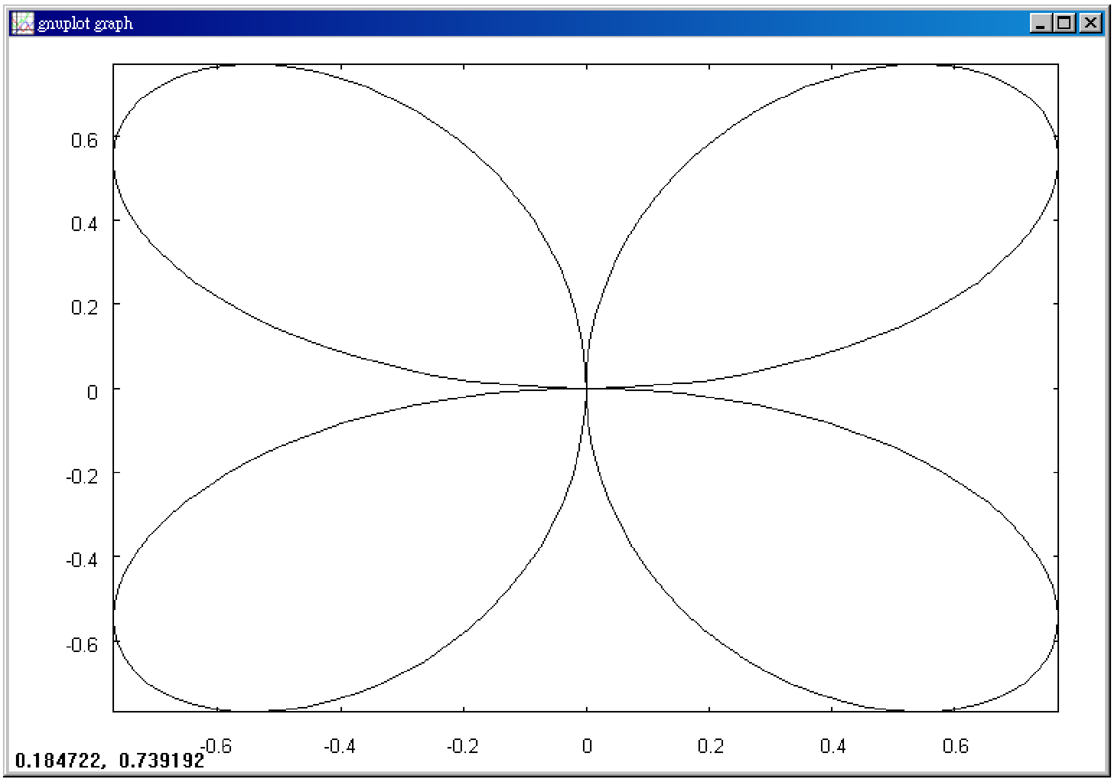
Sketch the curve $r = \sin 2\theta$.

Solution：(%i1) load(draw); 由於極坐標畫圖並沒有直接的指令，於是我們需要讀取模組 draw，此模組裡頭包含極坐標畫圖的指令 //讀取模組 draw

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp

(%i2) draw2d(nticks=200,polar(sin(2*theta),theta,0,2*%pi)); 極坐標畫圖指令：draw(nticks=點數，polar(函數式，角度，最小的角度，最大的角度)) //在此題畫極坐標表示的函數圖形爲 $\sin(2\theta)$，角度爲 θ，最小的角度是 0，最大的角度是 2π，用 200 個點描繪曲線

(%o2) [gr2d(polar)]



Step 1： $dr/d\theta = 2\cos 2\theta$.

Step 2： $r = 0$ at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. $dr/d\theta = 0$ at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Step 3： See Figure 7.8.8.

Step 4： We take values at intervals of $\frac{\pi}{8}$ beginning at $\theta = 0$. We can save some time by observing that the values from π to 2π are the same as those from 0 to π.

θ	$r = \sin 2\theta$	$dr/d\theta$	$\tan\psi$	Comments
0 and π	0	2	0	Crosses origin
$\pi/8$ and $9\pi/8$	$\sqrt{2}/2$	$\sqrt{2}$	1/2	$\|r\|$ increasing
$2\pi/8$ and $10\pi/8$	1	0	—	max
$3\pi/8$ and $11\pi/8$	$\sqrt{2}/2$	$-\sqrt{2}$	-1/2	$\|r\|$ decreasing
$4\pi/8$ and $12\pi/8$	0	-2	0	Crosses origin
$5\pi/8$ and $13\pi/8$	$-\sqrt{2}/2$	$-\sqrt{2}$	1/2	$\|r\|$ increasing
$6\pi/8$ and $14\pi/8$	-1	0	—	min
$7\pi/8$ and $15\pi/8$	$-\sqrt{2}/2$	$\sqrt{2}$	-1/2	$\|r\|$ decreasing

Step 5： We plot the points and trace out the curve as θ increases from 0 to 2π. Figure 7.8.9 shows the curve at various stages of development. The graph looks like a four-leaf clover.

If r approaches ∞ as θ approaches 0 or π, the curve may have a horizontal asymptote which can be found by computing the limit of y. At $\theta = \pi/2$ or $3\pi/2$ there may be vertical asymptotes. The method is illustrated in the following example.

Example 3.

Sketch $r = \tan(\frac{1}{2}\theta)$.

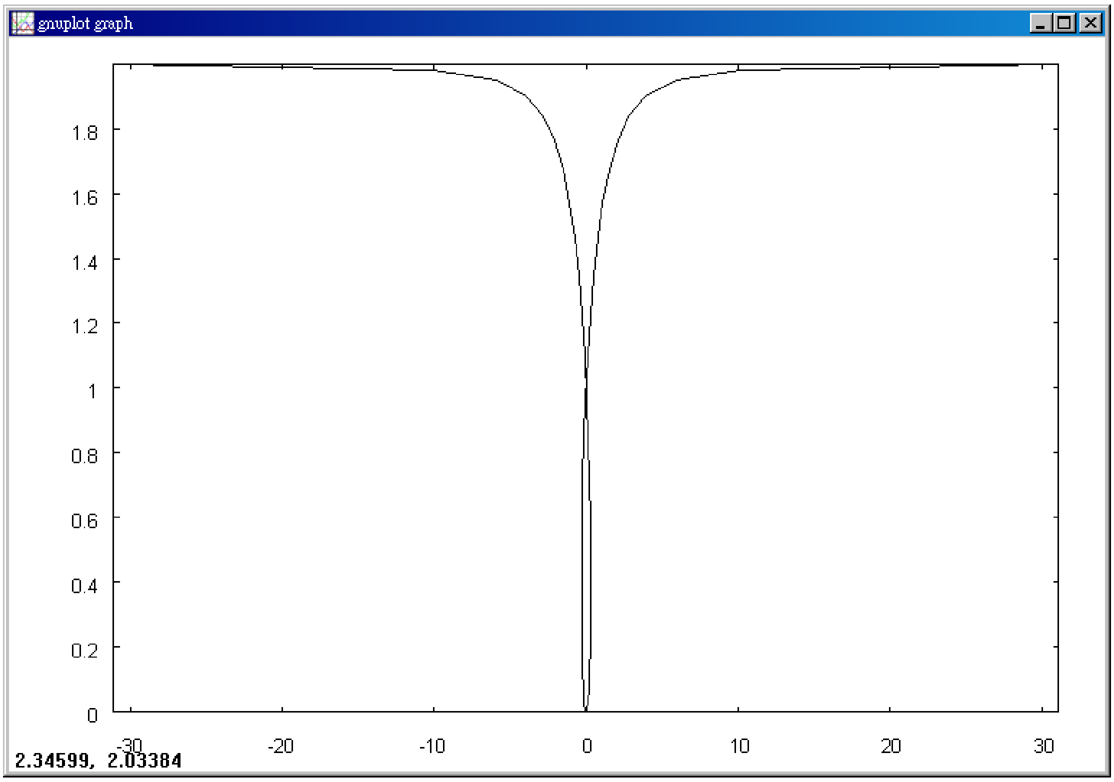
Solution：(%i1) load(draw); 由於極坐標畫圖並沒有直接的指令，於是我們需要讀取模組 draw，此模組裡頭包含極坐標畫圖的指令 //讀取模組 draw

```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp
```

(%i2) draw2d(nticks=50,polar(tan((1/2)*theta),theta,0,2*%pi)); 極坐標畫圖指令：

draw(nticks=點數，polar(函數式，角度，最小的角度，最大的角度))　//在此題畫極坐標表示的函數圖形爲 $\tan(\frac{1}{2}\theta)$ ，角度爲 θ ，最小的角度是 0，最大的角度是 2π ，用 50 個點描繪曲線

(%o2) [gr2d(polar)]



Step 1： $dr/d\theta = \frac{1}{2}\sec^2(\frac{1}{2}\theta)$.

$$y = r\sin\theta = \sin\frac{1}{2}\theta\sin\theta/\cos\frac{1}{2}\theta$$

$$= \sin\frac{1}{2}\theta(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta)/\cos\frac{1}{2}\theta = 2\sin^2(\frac{1}{2}\theta).$$

Step 2： $r = 0$ at $\theta = 0$.

r is undefined at $\theta = \pi$.

$dr/d\theta$ is never 0.

Step 3： See Figure 7.8.10.

Step 4：

θ	r or $\lim r$	$\lim y$	$dr/d\theta$	$\tan\psi$	Comments
0	0		1/2		crosses origin
$\pi/2$	1		1	1	$\lvert r\rvert$ increasing
$\theta \to \pi^-$	∞	2			asymptote $y=2$
$\theta \to \pi^+$	$-\infty$	2			asymptote $y=2$
$3\pi/2$	-1		1	-1	$\lvert r\rvert$ decreasing

Step 5： The curve crosses itself at the point $x=0,\quad y=1$, because this point has both polar coordinates

$(r=1,\quad \theta=\pi/2),\quad (r=-1,\quad \theta=3\pi/2)$.

Figure 7.8.11 shows the graph for various stages of development.

7.9 Area in Polar Coordinates

Example 1.

Find the area of one loop of the "four-leaf clover" $r=\sin 2\theta$.

Solution：(%i1) (1/2)*integrate(sin(2*theta)^2,theta,0,%pi/2); 積分指令：integrate(數式，變數，範圍) 函數爲 $\sin^2(2\theta)$，變數爲 θ，定積分範圍爲 0~$\pi/2$，z 積分算出來的數值後再乘上 1/2

(%o1) $\frac{\pi}{8}$

From Figure 7.9.6, we see that one loop is traced out then θ goes from 0 to $\pi/2$. Therefore the area is

$$A=\frac{1}{2}\int_0^{\pi/2}\sin^2(2\theta)d\theta=\frac{1}{2}\int_0^{\pi}\frac{1}{2}\sin^2\phi d\phi$$

$$=\frac{1}{4}\int_0^{\pi}\sin^2\phi d\phi=\frac{1}{4}(-\frac{1}{2}\sin\phi\cos\phi+\frac{1}{2}\phi)\Big|_0^{\pi}=\frac{1}{8}\pi.$$

As one would expect, all four loops have the same area.

On the loop form $\theta=\pi/2$ to $\theta=\pi$, the value of $r=\sin 2\theta$ is negative. However, the area is again

$$A=\frac{1}{2}\int_{\pi/2}^{\pi}\sin^2(2\theta)d\theta=\frac{1}{8}\pi.$$

Example 2.

Find the area of the region inside the circle $r=\sin\theta$ (Figure 7.9.7).

Solution：(%i1) integrate((1/2)*(sin(theta)^2),theta,0,%pi); 積分指令：integrate(數式，變數，範圍) 函數爲 $\frac{1}{2}\sin^2\theta$，變數爲 θ，定積分範圍爲 0~π

(%o1) $\frac{\pi}{4}$

The point (r,θ) goes around the circle once when $0\le\theta\le\pi$ with r positive, and again when $\pi\le\theta\le 2\pi$ with r negative. The theorem says that we will get the correct area if we take either 0 and π, or π and 2π, as the limits of integration. Thus

$$A=\int_0^{\pi}\frac{1}{2}\sin^2\theta d\theta=\frac{1}{2}(-\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta)\Big|_0^{\pi}=\frac{1}{4}(\pi-0)=\pi/4.$$

Alternatively,

$$A=\int_{\pi}^{2\pi}\frac{1}{2}\sin^2\theta d\theta=\frac{1}{2}(-\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta)\Big|_{\pi}^{2\pi}=\frac{1}{4}(2\pi-\pi)=\pi/4.$$

Since the curve is a circle of radius $\frac{1}{2}$, our answer $\pi/4$ agrees with the usual formula $A=\pi r^2$.

Integrating from 0 to 2π would count the area twice and give the wrong answer.

Example 3.

Find the area of the region inside both the circles $r=\sin\theta$ and $r=\cos\theta$.

Solution：(%i1) integrate((1/2)*(sin(theta)^2),theta,0,%pi/4)+integrate((1/2)*(cos(theta)^2),theta,%pi/4,%pi/2); 積分指令：integrate(數式，變數，範圍) 函數爲 $\frac{1}{2}\sin^2\theta$ 和 $\frac{1}{2}\cos^2\theta$，變數爲 θ，定積分範圍爲 $0\sim\pi/4$ 和 $\pi/4\sim\pi/2$，將兩積分後數值相加

(%o1) $$\frac{\frac{\pi}{4}-\frac{\pi+2}{8}}{2}+\frac{\pi-2}{16}$$

(%i2) factor(%); //用 factor 指令將上式做整理，%表示引用前式之結果

(%o2) $$\frac{\pi-2}{8}$$

The first thing to do is draw the graphs of both curves. The graphs are in Figure 7.9.8.

We see that the two circles intersect at the origin and at $\theta=\pi/4$. The region is divided into two parts, one bounded by $r=\sin\theta$ for $0\le\theta\le\pi/4$ and the other bounded by $r=\cos\theta$ for $\pi/4\le\theta\le\pi/2$. Thus

$$A=\int_{0}^{\pi/4}\frac{1}{2}\sin^2\theta d\theta+\int_{\pi/4}^{\pi/2}\frac{1}{2}\cos^2\theta d\theta$$

$$=\frac{1}{2}(-\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta)\Big|_{0}^{\pi/4}+\frac{1}{2}(\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta)\Big|_{\pi/4}^{\pi/2}$$

$$=\frac{1}{2}\left[(-\frac{1}{4}-0)+(\frac{\pi}{8}-0)+(0-\frac{1}{4})+(\frac{\pi}{4}-\frac{\pi}{8})\right]=\frac{\pi}{8}-\frac{1}{4}.$$

7.10 Length of A Curve in Polar Coordinates

Example 1.

Find the length of the spiral $r=\theta^2$ from $\theta=\pi$ to $\theta=4\pi$, shown in Figure 7.10.2.

Solution：(%i1) integrate(1/2*(sqrt(u)),u,(%pi)^2+4,16*(%pi)^2+4); 積分指令：integrate(數式，變數，範圍) 函數爲 $\frac{1}{2}\sqrt{u}$ ，變數爲 u，定積分範圍爲 $\pi^2+4\sim 16\pi^2+4$

(%o1) $$\frac{\frac{\sqrt{16\pi^2+4}\,(32\pi^2+8)}{3}-\frac{\sqrt{\pi^2+4}\,(2\pi^2+8)}{3}}{2}$$

$$s=\int_{\pi}^{4\pi}\sqrt{r^2+(dr/d\theta)^2}\,d\theta$$

$$=\int_{\pi}^{4\pi}\sqrt{\theta^4+4\theta^2}\,d\theta=\int_{\pi}^{4\pi}\sqrt{\theta^2+4}\,\theta d\theta.$$

Let $u=\theta^2+4,\quad du=2\theta d\theta$. Then

$$s=\int_{\pi^2+4}^{16\pi^2+4}\frac{1}{2}\sqrt{u}du=\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{\pi^2+4}^{16\pi^2+4}$$

$$=\frac{1}{3}((16\pi^2+4)^{3/2}-(\pi^2+4)^{3/2}).$$

Example 2.

Find the length of the curve $r = \sin\theta$ from $\theta = \alpha$ to $\theta = \beta$, shown in Figure 7.10.3. $dr/d\theta = \cos\theta$, so

$$s = \int_{\alpha}^{\beta} \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \int_{\alpha}^{\beta} d\theta = \beta - \alpha .$$

(%i1) integrate(sqrt(sin(theta)^2+cos(theta)^2),theta,alpha,beta); 積分指令：

integrate(數式，變數，範圍) 函數爲 $\sqrt{\sin^2\theta + \cos^2\theta}$ ，變數爲 θ ，定積分範圍爲 $\alpha \sim \beta$

```
Is  beta-α  positive, negative, or zero?positive;
Is  2 beta-π  positive, negative, or zero?positive;
Is  2 α-π  positive, negative, or zero?positive;
```

(%o1) $\operatorname{atan}\left(\frac{\sin(beta)}{\cos(beta)}\right) - \operatorname{atan}\left(\frac{\sin(\alpha)}{\cos(\alpha)}\right)$