6.1   INFINITE SUM THEOREM

Example 1   Find the volume of a pyramid of height h whose base has area B, as in Figure 6.1.7.

Place the pyramid on its side with the apex at $x = 0$ and the base at $x = h$. We use the fact that at any point x between 0 and h, the cross section has area proportional to x^2, so that

$$\frac{A(x)}{x^2} = \frac{B}{h^2},$$

$$A(x) = \frac{Bx^2}{h^2}$$

The volume is then

$$V = \int_0^h \frac{Bx^2}{h^2} dx = \frac{1}{3} \cdot \frac{Bx^3}{h^2} \Big]_0^h = \frac{1}{3} \cdot \frac{Bh^3}{h^2} = \frac{1}{3} Bh.$$

The solution is $V = (\frac{1}{3}) Bh.$

(%i2) integrate(b*x^2/h^2, x, 0, h);

(%o2) $\dfrac{b\,h}{3}$

◎ 函數為 $A(x) = \dfrac{Bx^2}{h^2}$,區間範圍從 **0** 到 **h,**可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

Example 2   A wedge is cut from a cylindrical tree trunk of radius 3 ft, by cutting the tree with two planes meeting on a line through the axis of the cylinder. The wedge is 1 ft thick at its thickest point. Find its volume.

The wedge is shown in Figure 6.1.8. The cross sections perpendicular to the x-axis are similar triangles. Place the edge along the x-axis with x from -3 to 3.At the thickest point, where $x = 0,$ the cross section is a triangle with base 3 ft and altitude 1 ft. The base of the cross section triangle at x is

$$b = \sqrt{9 - x^2},$$

and the altitude is

$$\frac{1}{3}b = \frac{1}{3}\sqrt{9 - x^2}.$$

The area of the cross section is

$$A(x) = \frac{1}{2} \cdot base \cdot altitude = \frac{1}{2}b \cdot \frac{1}{3}b = \frac{1}{6}(9 - x^2).$$

The volume is thus

$$V = \int_{-3}^{3} A(x)dx = \int_{-3}^{3} \frac{1}{6}(9 - x^2)dx = \left(\frac{3}{2}x - \frac{1}{18}x^3\right)\Big]_{-3}^{3} = 6\,ft^3.$$

The solution is 6 cubic feet.

(%i3) integrate((9-x^2)/6, x, -3, 3);

(%o3)　6

◎ 函數為 $A(x) = \frac{1}{6}(9 - x^2)$，區間範圍從-3 到 3，可使用指令

『integrate(函數, 變數, 範圍最小值, 範圍最大值); 』表示求積分

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6.2　　VOLUMES OF SOLIDS OF REVOLUTION

Example 1　　Find the volume of a right circular cone with height h and base of radius r . It is convenient to center the cone on the x -axis with its vertex at the origin as shown in Figure 6.2.5. This cone is the solid generated by rotating about the x -axis the triangular region R under the line $y = (r/h)x, 0 \le x \le h.$
Since x is the independent variable we use the Disc Method. The volume formula

gives $V = \int_0^h \pi(\frac{r}{h}x)^2 \ dx = \pi \frac{r^2}{h^2} \int_0^h x^2 \ dx = \pi \frac{r^2}{h^2} \frac{x^3}{3} \Big]_0^h = \frac{1}{3}\pi r^2 h,$

or $\qquad\qquad V = \frac{1}{3}\pi r^2 h .$

(%i2) integrate(%pi*((r*x/h))^2, x,0,h);

(%o2)　$\dfrac{\pi\,h\,r^2}{3}$

◎ 函數為 $y = \pi(\dfrac{r}{h}x)^2$, 區間範圍從 **0** 到 **h**,可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

Example 2　　The region R between the curves $y = 2 - x^2$ and $y = x^2$ is rotated about the x -axis generating a solid S. Find the volume of S.
The curves $y = 2 - x^2$ and $y = x^2$ cross at $x = \pm 1.$ The region is sketched in Figure 6.2.8. The volume is

$$V = \int_{-1}^{1} \pi(2 - x^2)^2 \ dx - \int_{-1}^{1} \pi(x^2)^2 \ dx$$

$$= \int_{-1}^{1} \pi(2 - x^2)^2 - \pi x^4 \ dx$$

$$= \int_{-1}^{1} \pi(4 - 4x^2)dx = \pi(4x - \frac{4}{3}x^3)\Big]_{-1}^{1} = 16\pi/3.$$

(%i3) integrate(%pi*((2-x^2))^2-%pi*((x^2))^2, x, -1, 1);

(%o3)　$\dfrac{16\,\pi}{3}$

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◎ 函數為 $y = \pi(2-x^2)^2 - \pi(x^2)^2$，區間範圍從 **-1** 到 **1**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

Example 3　　The region R between the line $y = 0$ and the curve $y = 2x - x^2$ is rotate about the y-axis to form a solid of revolution S. Find the volume of S. We use the cylindrical shell method because y is the dependent variable. We see that the curve crosses the x-axis at $x = 0$ and $x = 2$, and sketch the region in Figure 6.2.11. The volume is

$$V = \int_0^2 2\pi\, x(2x - x^2)dx = 2\pi \int_0^2 2x^2 - x^3\, dx = 2\pi\left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Bigg]_0^2 = \frac{8}{3}\pi.$$

(%i4) integrate(2*%pi*x*((2*x-x^2)), x, 0, 2);

$$(\%o4) \quad \frac{8\,\pi}{3}$$

◎ 函數為 $y = 2\pi\, x(2x - x^2)$，區間範圍從 **0** 到 **2**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

Example 4　　The region between the curves $y = x$ and $y = \sqrt{x}$ is rotated about the y-axis. Find the volume of the solid of revolution.

We make a sketch in Figure6.2.13 and find that the curves cross at $x = 0$ and $x = 1$. We take x for the independent variable and use the Cylindrical Shell Method.

$$V = \int_0^1 2\pi x(\sqrt{x} - x)dx = \int_0^1 2\pi(x^{3/2} - x^2)dx = 2\pi\left(\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right)\Bigg]_0^1 = \frac{2}{15}\pi \ .$$

(%i6) integrate(2*%pi*x*((sqrt((x))-x)), x, 0, 1);

$$(\%o6) \quad \frac{2\,\pi}{15}$$

◎ 函數為 $y = 2\pi x(\sqrt{x} - x)$，區間範圍從 **0** 到 **1**，可使用指令

『**integrate**(函數，變數，範圍最小值，範圍最大值);』表示求積分

Example 5   Derive the formula $V = \dfrac{4}{3}\pi r^3$ for the volume of a sphere by both the

Disc Method and the Cylindrical Shell Method.

The circle of radius r and center at the origin has the equation

$$x^2 + y^2 = r^2.$$

The region R inside this circle in the first quadrant will generate a hemisphere of radius r when it is rotated about the x-axis.

First take x as the independent variable and use the Disc Method. R is the region under the curve

$$y = \sqrt{r^2 - x^2}, \qquad 0 \le x \le r.$$

The hemisphere has volume

$$\frac{1}{2}V = \int_0^r \pi (f(x))^2 \, dx$$

$$= \int_0^r \pi (r^2 - x^2) \, dx = \pi r^2 x - \frac{1}{3}\pi x^3 \bigg]_0^r$$

$$= \pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3.$$

Therefore the sphere has volume

$$V = \frac{4}{3}\pi r^3$$

Now take y as the independent variable and use the Cylindrical Shell Method.

R is the region under the curve

$$x = \sqrt{r^2 - y^2}, \qquad 0 \le y \le r.$$

The hemisphere has volume

$$\frac{1}{2}V = \int_0^r 2\pi y\sqrt{r^2 - y^2} \, dy$$

Putting $u = r^2 - y^2$, $\quad du = -2y\,dy$, $\quad$ we get

$$\frac{1}{2}V = \int_{r^2}^{0} 2\pi\sqrt{u}\,(-\frac{1}{2})du = \int_{r^2}^{0} -\pi\sqrt{u}\,du = -\frac{2}{3}\pi\,u^{3/2}\,\Big]_{r^2}^{0}$$

Thus again $V = \frac{4}{3}\pi\,r^2$.

(%i7) integrate(%pi*((r^2-x^2)), x, 0, r);

(%o7) $\dfrac{2\,\pi\,r^3}{3}$

(%i9) solve[2*%o7];

(%o9) $solve$ $\dfrac{4\pi r^3}{3}$

◎ 函數為 $\pi(r^2 - x^2)$ ，區間範圍從 **0** 到 **r**，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

6.3   LENGTH OF A CURVE

Example 1   Find the length of the curve

$$y = 2x^{3/2}, \qquad 0 \le x \le 1$$

shown in Figure 6.3.3. We have

$$dy/dx = 3x^{1/2}, \qquad s = \int_0^1 \sqrt{1 + 9x}\, dx.$$

Put $u = 1 + 9x$. Then

$$s = \int_1^{10} \frac{1}{9}\sqrt{u}\, du = \frac{2}{3} \cdot \frac{1}{9} u^{3/2} \Big]_1^{10} = \frac{2}{27}(\sqrt{1000} - 1).$$

Sometimes a curve in the (x, y) plane is given by parametric equations

$$x = f(t), \qquad y = g(t), \qquad c \le t \le d$$

A natural example is the path of a moving particle where t is time. We give a formula for the length of such a curve.

(%i10) integrate(sqrt((1+9*x)), x, 0, 1);

$$(\%o10) \quad \frac{2 \cdot 10^{3/2}}{27} - \frac{2}{27}$$

◎ 函數為 $\sqrt{1+9x}$，區間範圍從 **0** 到 **1**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

Example 2   Find the length of the path of a ball whose motion is given by

$$x = 20t, \qquad y = 32t - 16t^2$$

from $t = 0$ until the ball hits the ground. (Ground level is $y = 0$, see Figure 6.3.5)
The ball is at ground level when

$$32t - 16t^2 = 0 \quad, \quad t = 0 \quad \text{and} \quad t = 2.$$

We have $dx/dt = 20, \qquad dy/dt = 32 - 32t,$

$$s = \int_0^2 \sqrt{20^2 + (32 - 32t)^2}\, dt.$$

We cannot evaluate this integral yet, so the answer is left in the above form.

We can get an approximate answer by the Trapezoidal Rule. When $\Delta x = \dfrac{1}{5}$,

the Trapezoidal Approximation is

$$s \sim 53.5 \qquad \text{error} \le 0.4$$

(%i14) diff(20*t,t,1);

(%o14) 20

(%i15) diff(32*t-16*t^2,t,1);

(%o15) 32 - 32 t

(%i11) integrate(sqrt((20^2+((32-32*t))^2)), t, 0, 2);

(%o11) $\dfrac{25\, \text{asinh}\left(\dfrac{8}{5}\right) + 8\sqrt{89}}{2}$

(%i13) float(%), numer;

(%o13) $solve_{53.34821612703737}$

◎ 函數為 $\sqrt{20^2 + (32-32t)^2}$ ，區間範圍從 **0** 到 **2**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

Example 3　　Let

$$x = 1 - t^2, \qquad y = 1, \qquad -1 \le t \le 1.$$

As t goes from -1 to 1, the point (x, y) moves from (0, 1) to (1, 1) and then back along the same line to (0, 1) again. The path is shown in Figure 6.3.6.

The path has length one. However, the point goes along the path twice for a total distance of two. The length formula gives the total distance the point moves.

$$s = \int_{-1}^{1} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{-1}^{1} \sqrt{(-2t)^2 + 0^2}\, dt$$

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$$= \int_{-1}^{1} \sqrt{4t^2}\, dt = \int_{-1}^{1} 2|t|\, dt = 2$$

(%i16) diff(1-t^2,t,1);

  (%o16) $-2\, t$

(%i17) diff(1,t,1);

  (%o17) 0

(%i18) integrate(sqrt(((-2*t)^2)), t, -1, 1);

  (%o18) $2\displaystyle\int_{-1}^{1} |t|\, d\, t$

◎ 函數為 $\sqrt{4t^2}$ ，區間範圍從 **-1** 到 **1**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

6.4   AREA OF A SURFACE OF REVOLUTION

Example 1   The line segment $y = 3x$, from $x = 1$ to $x = 4$, is rotated about the y-axis (Figure 6.4.7). Find the area of the surface of revolution.

FIRST SOLUTION   We use the integration formula. $dy / dx = 3$, so

$$A = \int_1^4 2\pi \, x \sqrt{1 + (dy/dx)^2} \, dx$$

$$= \int_1^4 2\pi \, x \sqrt{1 + 3^2} \, dx = 2\pi \sqrt{10} \int_1^4 x \, dx$$

$$= 2\pi \sqrt{10} \frac{x^2}{2} \Bigg]_1^4 = 2\pi \sqrt{10} \left(\frac{16-1}{2} \right) = 15\pi \sqrt{10}$$

SECOND SOLUTION   This surface of revolution is a frustum of a cone, so the formula for the lateral area of a frustum can be used directly. From the diagram we that the radii and slant height are :

$$r_1 = 1, \qquad r_2 = 4,$$

$$l = \text{distance from (1,3) to (4,12)}$$

$$= \sqrt{(4-1)^2 + (12-3)^2} = \sqrt{3^2 + 9^2} = \sqrt{90} = 3\sqrt{10}.$$

Then $A = \pi (r_1 + r_2)l = \pi(1+4)3\sqrt{10} = 15\pi \sqrt{10}$

(%i21) diff(3*x,x,1);

(%o21) 3

(%i22) integrate(2*%pi*x*sqrt((1+3^2)), x, 1, 4);

(%o22) $15\sqrt{10}\,\pi$

◎ 函數為 $2\pi \, x \sqrt{1 + (dy/dx)^2}$ ，區間範圍從 **1** 到 **4**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

Example 2   The curve $y = \dfrac{1}{2}x^2$, $0 \le x \le 1$, is rotated about the y-axis (Figure 6.4.8). Find the area of the surface of revolution.

$$\frac{dy}{dx} = x,$$

$$A = \int_0^1 2\pi\, x\sqrt{1 + (dy/dx)^2}\, dx$$

$$= \int_0^1 2\pi\, x\sqrt{1 + x^2}\, dx = \int_1^2 \pi\sqrt{u}\, du \qquad \text{(where } u = 1 + x^2\text{)}$$

$$= \frac{2}{3}\pi\, u^{3/2} \Bigg]_1^2 = \frac{2}{3}\pi(2\sqrt{2} - 1).$$

(%i23) diff(x^2/2,x,1);

(%o23) x

(%i24) integrate(2*%pi*x*sqrt((1+x^2)), x, 0, 1);

(%o24) $2\left(\dfrac{2^{3/2}}{3} - \dfrac{1}{3}\right)\pi$

◎ 函數為 $2\pi\, x\sqrt{1 + x^2}$ ，區間範圍從 **1** 到 **2**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

Example 3   The curve $x = 2t^2$, $y = t^3$, $0 \le t \le 1$ is rotated about the y-axis. Find the area of the surface of revolution (Figure 6.4.11).

We first find dx/dt and dy/dt and then apply the formula for area.

$$\frac{dx}{dt} = 4t, \qquad \frac{dy}{dt} = 3t^2,$$

$$A = \int_0^1 2\pi\, x\sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$$

$$= \int_0^1 4\pi\, t^2\sqrt{(4t)^2 + (3t^2)^2}\, dt$$

$$= 4\pi \int_0^1 t^2 \sqrt{16t^2 + 9t^4}\,dt$$

$$= 4\pi \int_0^1 t^3 \sqrt{16 + 9t^2}\,dt$$

Let $u = 16 + 9t^2$, $du = 18t\,dt$, $dt = \dfrac{1}{18t}\,du$, $t^2 = \dfrac{u - 16}{9}$. Then $u = 16$ at $t = 0$

and $u = 25$ at $t = 1$, so

$$A = 4\pi \int_{16}^{25} t^3 \sqrt{u}\,\frac{1}{18t}\,du = 4\pi \int_{16}^{25} \frac{1}{18}\left(\frac{u - 16}{9}\right)\sqrt{u}\,du$$

$$= \frac{2\pi}{81} \int_{16}^{25} (u^{3/2} - 16\sqrt{u})\,du = \frac{5692}{1215}\pi \sim 4.7\pi\,.$$

(%i25) diff(2*t^2,t,1);

  (%o25) 4 t

(%i26) diff(t^3,t,1);

  (%o26) 3 t^2

(%i27) integrate(4*%pi*t^2*sqrt(((((4*t)^2)+((3*t^2))^2))), t, 0, 1);

  (%o27) $\dfrac{5692\,\pi}{1215}$

(%i28) float(%), numer;

  (%o28) 14.71765052200255

◎ 函數為 $4\pi\,t^2\sqrt{(4t)^2 + (3t^2)^2}$ ，區間範圍從 **0** 到 **1**，可使用指令

『**integrate(** 函數，變數，範圍最小值，範圍最大值 **);**』表示求積分

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Example 4   Derive the formula $A = 4\pi r^2$ for the area of the surface of a sphere of radius r .

When the portion of the circle $x^2 + y^2 = r^2$ in the first quadrant is rotated about the $y-$ axis it will form a hemisphere of radius r (Figure 6.4.12). The surface of the sphere has twice the area of this hemisphere.

It is simpler to take y as the independent variable, so the curve has the equation

$$x = \sqrt{r^2 - y^2}, \qquad 0 \le y \le r .$$

then $\dfrac{dx}{dy} = -\dfrac{y}{\sqrt{r^2 - y^2}} .$

This derivative is undefined at $y = 0$. To get around this difficulty we let $0 < a < r$ and divide the surface into the two parts shown in Figure 6.4.13, generated by the curve from $y = a$ to $y = r$.

The area of C is

$$C = \int_a^r 2\pi\, x \sqrt{(dx/dy)^2 + 1}\; dy$$

$$= \int_a^r 2\pi \sqrt{r^2 - y^2}\, \sqrt{1 + y^2/(r^2 - y^2)}\; dy$$

$$= \int_a^r 2\pi \sqrt{r^2 - y^2}\, \sqrt{r^2/(r^2 - y^2)}\; dy$$

$$= \int_a^r 2\pi\, r\, dy = 2\pi\, r\Big]_a^r = 2\pi\, r(r - a).$$

We could find the area of B by taking x as the independent variable. However, it is simpler to let a be an infinitesimal ε . Then B is an infinitely thin ring-shaped surface, so its area is infinitesimal. Therefore the hemisphere has area

$$\frac{1}{2}A = B + C \approx 0 + 2\pi\, r(r - \varepsilon) \approx 2\pi\, r^2,$$

or $\dfrac{1}{2}A = 2\pi\, r^2$

and the sphere has $A = 4\pi\, r^2$.

◎ 函數為 $2\pi \sqrt{r^2 - y^2}\, \sqrt{1 + y^2/(r^2 - y^2)}$ ，區間範圍從 **a** 到 **r**，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

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(%i29) integrate(2*%pi*sqrt(r^2-y^2)*sqrt(1+y^2/(r^2-y^2)), y, a, r);

```
Is   r   positive or negative?positive;
```

$$(\%o29) \quad 2\,\pi\,(r^2 - a\,r)$$

Example 5   Let C be the curve
$$y = x^4, \qquad 0 \le x \le 1.$$
Set up an integral for the surface area generated by rotating the curve C about
(a) the y-axis, (b) the x-axis

(a) $dy/dx = 4x^3$

$$A = \int_0^1 2\pi\, x \sqrt{1 + (dy/dx)^2}\, dx$$

$$= \int_0^1 2\pi\, x \sqrt{1 + 16x^6}\, dx$$

We cannot evaluate this integral, so we leave it in the above it in the above form. The

Trapezoidal Rule can be used to get approximate values. When $\Delta x = \dfrac{1}{10}$ the

Trapezoidal Approximation is
$$A \sim 6.42, \quad error \le 0.26.$$

◎ 函數為 $2\pi\, x\sqrt{1 + 16x^6}$ ，區間範圍從 **0** 到 **1**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i31) integrate(2*%pi*x*sqrt((1+16*x^6)), x, 0, 1);

$$(\%o31) \quad 2\,\pi \int_0^1 x\sqrt{16\,x^6 + 1}\; dx$$

(b) $$A = \int_0^1 2\pi\, y \sqrt{1 + (dy/dx)^2}\, dx$$

$$= \int_0^1 2\pi\, x^4 \sqrt{1 + 16x^6}\, dx.$$

The Trapezoidal Approximation when $\Delta x = \dfrac{1}{10}$ is

$$A \sim 3.582, \qquad error \leq 0.9.$$

◎ 函數為 $2\pi\, x^4 \sqrt{1+16x^6}$ ，區間範圍從 **0** 到 **1**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i33) integrate(2*%pi*x^4*sqrt((1+16*x^6)), x, 0, 1);

(%o33) $2\pi \displaystyle\int_0^1 x^4 \sqrt{16x^6+1}\, dx$

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6.5   AVERAGES

Example 1   Find the average value of $y = \sqrt{x}$ from $x = 1$ to $x = 4$

$$y_{ave} = \frac{\int_1^4 \sqrt{x}\, dx}{(4-1)} = \frac{\frac{2}{3}x^{3/2}\Big]_1^4}{3} = \frac{\frac{2}{3}(8-1)}{3} = \frac{14}{9}$$

◎ 函數爲 $\dfrac{\int_1^4 \sqrt{x}\, dx}{(4-1)}$ ，區間範圍從 **1** 到 **4**，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大値 **);**』表示求積分

(%i34) integrate(sqrt((x))/3, x, 1, 4);

(%o34) $\dfrac{14}{9}$

Example 2   A car starts at rest and moves with velocity $v = 3t^2$. Find its average velocity between times $t = 0$ and $t = 5$. At what point of time is its velocity equal to the average velocity?

$$v_{ave} = \frac{\int_0^5 3t^2\, dt}{5-0} = \frac{t^3\big]_0^5}{5} = \frac{125}{5} = 25.$$

To find the value of t where $v = v_{ave}$, we put

$$3t^2 = 25, \qquad t = \sqrt{25/3} = 5/\sqrt{3}.$$

◎ 函數爲 $\dfrac{\int_0^5 3t^2\, dt}{5-0}$ ，區間範圍從 **0** 到 **5**，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大値 **);**』表示求積分

(%i36) integrate(3*t^2/5-0, t, 0, 5);

(%o36) 25

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Example 3 　　A car travels with velocity $v = 4t + 10$, where t is time. Between times $t = 0$ and $t = 4$ find the average velocity with respect to (a) time, and (b) distance.

(a) 　　　　　　　　$v_{ave} = \dfrac{\int_0^4 4t + 10\, dt}{4} = \dfrac{2t^2 + 10t\big]_0^4}{4} = 18$

◎ 函數為 $\dfrac{\int_0^4 4t + 10\, dt}{4}$ ，區間範圍從 **0** 到 **4**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i37) integrate((4*t+10)/4, t, 0, 4);

(%o37)　18

(b) Let s be the distance, and put $s = 0$ when $t = 0$. Since $ds = v\, dt = (4t + 10)\, dt$, at time $t = 4$ we have

$$s = \int_0^4 4t + 10\, dt = 2t^2 + 10t\Big]_0^4 = 72.$$

Then 　　　　$v_{ave} = \dfrac{\int_0^{72} (4t + 10)\, ds}{72} = \dfrac{\int_0^4 (4t + 10)(4t + 10)\, dt}{72}$

$$= \dfrac{\int_0^4 16t^2 + 80t + 100\, dt}{72} = \dfrac{\frac{16}{3}t^3 + 40t^2 + 100t\big]_0^4}{72}$$

$$= \dfrac{1024/3 + 640 + 400}{72} \sim 19.2$$

(%i39) integrate((4*t+10)/72, s, 0, 72);

(%o39)　4 t + 10

(%i40) integrate((4*t+10)*(4*t+10)/72, t, 0, 4);

(%o40) $\dfrac{518}{27}$

(%i41) float(%), numer;

(%o41) 19.18518518518519

◎ 函數為 $\dfrac{\int_0^{72}(4t+10)\,ds}{72}$ ，區間範圍從 **0** 到 **72,** 可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

6.6   SOME APPLICATIONS TO PHYSICS

Example 1   Find the mass of a wire 6 cm long whose density at distance x from the center is $9 - x^2$ gm/cm. In Figure6.6.2, we put the center of the wire at the origin.

Then $\qquad m = \int_{-3}^{3} 9 - x^2 \, dx = 9x - \frac{1}{3}x^3 \Big]_{-3}^{3} = 36 \, gm.$

◎ 函數為　$9 - x^2$，區間範圍從 **-3** 到 **3**，可使用指令

『**integrate(函數，變數，範圍最小值，範圍最大值);**』表示求積分

(%i42) integrate(9-x^2, x, -3, 3);

(%o42) 36

Example 2   A circular disc of radius r has density at each point equal to the distance of the point from the $y-$axis. Find its mass. The circle is the region

between the curves $-\sqrt{r^2 - x^2}$ and $\sqrt{r^2 - x^2}$ from $-r$ to r. The density at a

point (x, y) in the disc is $|x|$. By symmetry, all four quadrants have the same mass.

We shall find the mass m_1 of the first quadrant and multiply by four.

$$m_1 = \int_0^r \sqrt{r^2 - x^2} \; x \, dx.$$

Put $u = r^2 - x^2, \;\; du = -2x \, dx; \;\; u = r^2$ when $x = 0,$ and $u = 0$ when $x = r.$

$$m_1 = \int_{r^2}^{0} -\frac{1}{2}\sqrt{u} \, du = \frac{1}{2}\int_0^{r^2} \sqrt{u} \, du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} \Big]_0^{r^2} = \frac{1}{3}r^3.$$

Then $\;\; m = 4m_1 = \frac{4}{3}r^3.$

◎ 函數為 $\sqrt{r^2-x^2}\,x$ ，區間範圍從 **0** 到 **r**，可使用指令

『**integrate(** 函數**,** 變數**,** 範圍最小值**,** 範圍最大值 **);**』表示求積分

(%i43) integrate(sqrt((r^2-x^2))*x, x, 0, r);

Is   r   positive, negative, or zero?positive;

(%o43) $\dfrac{r^3}{3}$

Example 3   A wire between $x = 0$ and $x = 1$ has density $\rho(x) = x^2$

The moment is $M = \int_0^1 x^2 x\, dx = \dfrac{x^4}{4}\bigg]_0^1 = \dfrac{1}{4}.$

◎ 函數為 $x^2\,x$ ，區間範圍從 **0** 到 **1**，可使用指令

『**integrate(** 函數**,** 變數**,** 範圍最小值**,** 範圍最大值 **);**』表示求積分

The mass and center of mass are

$$m = \int_0^1 x^2\, dx = \frac{1}{3},$$

$$\bar{x} = M / m = \frac{3}{4}.$$

◎ 函數為 x^2 ，區間範圍從 **0** 到 **1**，可使用指令

『**integrate(** 函數**,** 變數**,** 範圍最小值**,** 範圍最大值 **);**』表示求積分

(%i44) integrate(x^2, x, 0, 1);

(%o44) $\dfrac{1}{3}$

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(%i45) integrate(x^2*x, x, 0, 1);

(%o45) $\frac{1}{4}$

(%i47) solve[(1/4)/(1/3)];

(%o47) $solve_{3/4}$

Example 4　　Find the centroid of the triangular region R　bounded by the $x-axis$,

the $y-axis$, and the line $y = 1 - \frac{1}{2}x$　shown in Figure 6.611. R　is the region under

the curve $y = 1 - \frac{1}{2}x$　from $x = 0$　to $x = 2$. The area of R　is

$$A = \int_0^2 1 - \frac{1}{2}x\,dx = x - \frac{1}{4}x^2 \bigg]_0^1 = 1.$$

◎ 函數為 $1 - \frac{1}{2}x$ ，區間範圍從 **0** 到 **2**，可使用指令
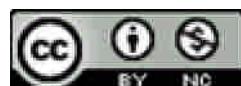
『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

(%i48) integrate(1-x/2, x, 0, 2);

(%o48) 1

The centroid is $(\bar{x}, \bar{y})$　where

$$\bar{x} = \int_0^2 x(1 - \frac{1}{2}x)dx = \frac{1}{2}x^2 - \frac{1}{6}x^3 \bigg]_0^2 = \frac{2}{3},$$

◎ 函數為 $x(1-\frac{1}{2}x)$ ，區間範圍從 **0** 到 **2**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i49) integrate((1-x/2)*x, x, 0, 2);

(%o49) $\dfrac{2}{3}$

$$\overline{y} = \int_0^2 \frac{1}{2}(1-\frac{1}{2}x)^2 \, dx = \int_0^2 \frac{1}{2} - \frac{1}{2}x + \frac{1}{8}x^2 \, dx$$

$$= \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{24}x^3 \Big]_0^2 = \frac{1}{3}.$$

Thus the centroid is the point $(\frac{2}{3}, \frac{1}{3})$.

◎ 函數為 $\frac{1}{2}(1-\frac{1}{2}x)^2$ ，區間範圍從 **0** 到 **2**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i51) integrate((1-x/2)^2/2, x, 0, 2);

(%o51) $\dfrac{1}{3}$

Example 5 Find the centroid of the semicircle $y = \sqrt{1-x^2}$. By symmetry, the

centroid is on the $y-$axis, $\overline{x} = 0$. The area of the semicircle is $\frac{1}{2}\pi$.

Then

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$$\frac{1}{2}\pi\,\bar{y} = \int_{-1}^{1}\frac{1}{2}(1-x^2)\,dx = \frac{1}{2}x - \frac{1}{6}x^3\bigg]_{-1}^{1} = \frac{2}{3},$$

$$\bar{y} = \frac{2/3}{\frac{1}{2}\pi} = \frac{4}{3\pi}.$$

◎ 函數為 $\dfrac{1}{2}(1-x^2)$ ，區間範圍從 **-1** 到 **1**，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

(%i52) integrate((1-x^2)/2, x, -1, 1);

(%o52) $\dfrac{2}{3}$

Example 6   A semicircular plate of radius one, constant density, and mass m lies flat on the table. (a)How much work is required to stand it up with the straight edge horizontal on the table? (b)How much work is required to stand it up with the straight edge vertical and one corner on the table? From the pervious exercise, we know that the center of mass is on the central radius $4/3\pi$ from the center of the circle. Put the $x-$axis on the surface of the table.

(a) The center of mass is lifted a distance $4/3\pi$ above the table. Therefore
$W = mg \cdot 4/3\pi$.

(b) The center of mass is lifted a distance 1 above the table, so $W = mg$.

Example 7   A spring shown in Figure 6.6.15, of natural length L exerts a force $F = cx$ when compressed a distance x. Find the work done in compressing the spring from length $L-a$ to length $L-b$.

$$W = \int_{a}^{b} cx\,dx = \frac{1}{2}cx^2\bigg]_{a}^{b} = \frac{1}{2}c(b^2 - a^2).$$

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◎ 函數為 cx ，區間範圍從 **a** 到 **b**，可使用指令

『**integrate（** 函數，變數，範圍最小值，範圍最大值 **）；**』表示求積分

(%i53) integrate(c*x, x, a, b);

$$(\%o53) \quad \left(\frac{b^2}{2} - \frac{a^2}{2} \right) c$$

Example 8   The force of gravity between two particles of mass m_1 and m_2 is
$$F = gm_1 m_2 / s^2,$$
where g is a constant and s is the distance between the particles. Find the work required to move the particle m_2 from a distance a to a distance b from m_1.
$$W = \int_a^b F\, ds = \int_a^b \frac{gm_1 m_2}{s^2} ds = gm_1 m_2 (-s^{-1}) \Big]_a^b = gm_1 m_2 \left(\frac{1}{a} - \frac{1}{b} \right).$$

◎ 函數為 $\dfrac{gm_1 m_2}{s^2}$ ，區間範圍從 **a** 到 **b**，可使用指令

『**integrate（** 函數，變數，範圍最小值，範圍最大值 **）；**』表示求積分

(%i55) integrate((g*m1*m2)/s^2, s, a, b);

```
Is   b-a   positive, negative, or zero?positive;
Is   b   positive, negative, or zero?positive;
Is   a   positive, negative, or zero?positive;
```

$$(\%o55) \quad \left(\frac{1}{a} - \frac{1}{b} \right) g\, m1\, m2$$

6.7 IMPROPER INTEGRALS

Example 1   Find $\int_0^1 \frac{1}{\sqrt{x}}dx$. For $u > 0$,

$$\int_u^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big]_u^1 = 2 - 2\sqrt{u}.$$

Then $\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{u \to 0^+}\int_u^1 \frac{1}{\sqrt{x}}dx = \lim_{u \to 0^+}(2 - 2\sqrt{u}) = 2.$

Therefore the region under the curve $y = 1/\sqrt{x}$ from 0 to 1 shown in Figure 6.7.1

has area 2, and the improper integral converges.

◎ 函數爲 $\frac{1}{\sqrt{x}}$ ，區間範圍從 **0** 到 **1**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i57) integrate(1/sqrt((x)), x, 0, 1);

(%o57) 2

Example 2   Find $\int_0^1 x^{-2}dx$. For $u > 0$,

$$\int_u^1 x^{-2}dx = -x^{-1}\Big]_u^1 = -1 + \frac{1}{u}$$

This time $\lim_{u \to 0^+}\int_u^1 x^{-2}dx = \lim_{u \to 0^+}(-1 + \frac{1}{u}) = \infty.$

The improper integral diverges. Since the limit goes to infinity we may write

$$\int_0^1 x^{-2}dx = \infty.$$

The region under the curve in Figure 6.7.2 is said to have *infinite area*.

◎ 函數爲 x^{-2} ，區間範圍從   **0** 到   **1**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

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(%i58) integrate(x^((-2)), x, 0, 1);

```
defint: integral is divergent.
 -- an error.  To debug this try debugmode(true);
```

(%i76) limit(x^((-2)), x, 0, plus);

(%o76) ∞

Example 3　　Find the length of the curve　$y = x^{2/3}$,　$0 \le x \le 8$.　From Figure 6.7.3 the curve must have finite length. However, the derivative

$$\frac{dy}{dx} = \frac{2}{3} x^{-1/3}$$

is undefined at　$x = 0$.　Thus the length formula gives an improper integral,

$$s = \int_0^8 \sqrt{1 + (dy/dx)^2}\, dx = \int_0^8 \sqrt{1 + \frac{4}{9} x^{-2/3}}\, dx = \int_0^8 \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}}\, dx$$

$$= \int_0^8 \frac{1}{3x^{1/3}} \sqrt{9x^{2/3} + 4}\, dx = \lim_{a \to 0^+} \int_a^8 \frac{1}{3x^{1/3}} \sqrt{9x^{2/3} + 4}\, dx.$$

Let　$u = 9x^{2/3} + 4,\ \ du = 6x^{-1/3}\, dx$. The indefinite integral is

$$\int \frac{1}{3x^{1/3}} \sqrt{9x^{2/3} + 4}\, dx = \int \frac{1}{18} \sqrt{u}\, du = \frac{1}{18} \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{1}{27} (9x^{2/3} + 4)^{3/2} + C.$$

Therefore　　　　　　$s = \lim_{a \to 0^+} \frac{1}{27} (9x^{2/3} + 4)^{3/2} \Big]_a^8$

$$= \frac{1}{27} ((9 \cdot 4 + 4)^{3/2} - (9 \cdot 0 + 4)^{3/2}) = \frac{8}{27} (10\sqrt{10} - 1).$$

◎ 函數為　$\sqrt{1 + \frac{4}{9} x^{-2/3}}$　，區間範圍從　**0** 到　**8**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i59) diff(x^((2/3)),x,1);

(%o59) $\dfrac{2}{3\,x^{1/3}}$

(%i60) integrate(sqrt((1+4/((9*x^((2/3))))))), x, 0, 8);

(%o60) $\dfrac{2^{9/2}\,5^{3/2}}{27} - \dfrac{8}{27}$

Example 4   Find the area under the curve $y = x^{-3}$ from 1 to ∞. The area is given by the improper integral

$$\int_1^{\infty} x^{-3}\,dx.$$

For $u > 0$, $\displaystyle\int_1^u x^{-3}\,dx = -\frac{1}{2}x^{-2}\bigg]_1^u = -\frac{1}{2}u^{-2} + \frac{1}{2}.$

Thus $\displaystyle\int_1^{\infty} x^{-3}\,dx = \lim_{u\to\infty}\int_1^u x^{-3}\,dx = \lim_{u\to\infty}(-\frac{1}{2}u^{-2} + \frac{1}{2}) = \frac{1}{2}.$

◎ 函數為　x^{-3}　，區間範圍從　**1** 到　∞　，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

(%i61) integrate(x^((-3)), x, 1, inf);

(%o61) $\dfrac{1}{2}$

Example 5   Find the area under the curve $y = x^{-2/3}$, $1 \le x < \infty$

$$A = \int_1^{\infty} x^{-2/3}\,dx = \lim_{u\to\infty}\int_1^u x^{-2/3}\,dx$$

$$= \lim_{u\to\infty} 3x^{1/3}\bigg]_1^u = \lim_{u\to\infty} 3(u^{1/3} - 1) = \infty.$$

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◎ 函數為 $x^{-2/3}$ ，區間範圍從 **1** 到 ∞ ，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i79) integrate(x^((-2/3)), x, 1, u);

Is u−1 positive, negative, or zero?positive;

(%o79) $3\,u^{1/3}-3$

(%i80) limit(3*u^(1/3)-3, u,inf);

(%o80) ∞

Example 6　The region in Example 5 is rotated about the x-axis. Find the volume of the solid of revolution.

We use the Disc Method because the rotation is about the axis of the independent variable. The volume formula gives us an improper integral.

$$V = \int_1^\infty \pi\,(x^{-2/3})^2\,dx = \int_1^\infty \pi\,x^{-4/3}\,dx$$

$$= \lim_{u\to\infty}\int_1^\infty \pi\,x^{-4/3}\,dx = \lim_{u\to\infty} -3\pi\,x^{-1/3}\Big]_1^u$$

$$= \lim_{u\to\infty} 3\pi(-u^{-1/3}+1) = 3\pi$$

So the solid shown in Figure 6.7.5 has finite volume $V = 3\pi$.

◎ 函數為 $\pi(x^{-2/3})^2$ ，區間範圍從 **1** 到 ∞ ，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i63) integrate(%pi*(x^((-2/3)))^2, x, 1, inf);

(%o63) $3\,\pi$

Example 7   Find the improper integral $\int_{-8}^{1} x^{-1/3}\,dx.$ $x^{-1/3}$ is discontinuous at $x = 0$.

The indefinite integral is

$$\int x^{-1/3}\,dx = \frac{3}{2}x^{2/3} + C.$$

Then

$$\int_{-8}^{0} x^{-1/3}\,dx = \lim_{u\to 0^-}\int_{-8}^{u} x^{-1/3}dx = \lim_{u\to 0^-}\frac{3}{2}x^{2/3}\Bigg]_{-8}^{u}$$

$$= \lim_{u\to 0^-}(\frac{3}{2}u^{2/3} - \frac{3}{2}(-8)^{2/3}) = -\frac{3}{2}\cdot 4 = -6.$$

◎ 函數為　$x^{-1/3}$，區間範圍從　**-8** 到　**0**，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

(%i64) integrate(x^((-1/3)), x, -8, 0);

(%o64)　-6

Similarly,

$$\int_{0}^{1} x^{-1/3}\,dx = \frac{3}{2}.$$

So

$$\int_{-8}^{1} x^{-1/3}dx = -6 + \frac{3}{2} = -\frac{9}{2}$$

◎ 函數為　$x^{-1/3}$，區間範圍從 **0**　到　**1**，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

(%i65) integrate(x^((-1/3)), x, 0, 1);

(%o65)　$\dfrac{3}{2}$

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(%i66) solve[-6+3/2];

(%o66) $solve_{-\frac{9}{2}}$

Example 8   Find $\int_0^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx.$

The function $2/\sqrt{x} + 1/\sqrt{2-x}$ is continuous the open interval (0, 2) but discontinuous at both endpoints. Thus

$$\int_0^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx = \int_0^1 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx + \int_1^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx.$$

First we find the indefinite integral.

$$\int \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx = 4\sqrt{x} - 2\sqrt{2-x} + C.$$

Then $\int_0^1 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx = \lim_{u \to 0^+} \int_u^1 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx$

$$= \lim_{u \to 0^-} (4\sqrt{x} - 2\sqrt{2-x}) \Big]_u^1$$

$$= (4\text{-}2)\text{-}(0\text{-} 2\sqrt{2}) = 2 + 2\sqrt{2}.$$

Also $\int_1^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx = \lim_{v \to 2^-} \int_1^v \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx$

$$= \lim_{v \to 2^-} (4\sqrt{x} - 2\sqrt{2-x}) \Big]_1^v$$

$$= (4\sqrt{2} - 0)\text{-}(4\text{-}2) = 4\sqrt{2} - 2.$$

Therefore $\int_0^2 \frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}} dx = (2 + 2\sqrt{2}) + (4\sqrt{2} - 2) = 6\sqrt{2}.$

◎ 函數為 $\frac{2}{\sqrt{x}} + \frac{1}{\sqrt{2-x}}$ ，區間範圍從   **0** 到   **2**，可使用指令

『**integrate(** 函數, 變數, 範圍最小值, 範圍最大值 **);**』表示求積分

(%i70) integrate(2/sqrt((x))+1/sqrt((2-x)), x, 0, 1);

(%o70) $2^{3/2} + 2$

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(%i71) integrate(2/sqrt((x))+1/sqrt((2-x)), x, 1, 2);

(%o71) $2^{5/2} - 2$

(%i73) solve[%o70+%o71];

(%o73) $solve_{2^{5/2}+2^{3/2}}$

Example 9   Find $\int_0^1 \dfrac{1}{x^2} + \dfrac{1}{(x-1)^2}\, dx$.

The function $1/x^2 + 1/(x-1)^2$ is continuous on the open interval (0, 1) but discontinuous at both endpoints. The indefinite integral is

$$\int \frac{1}{x^2} + \frac{1}{(x-1)^2}dx = -\frac{1}{x} - \frac{1}{x-1} + C..$$

We have $\displaystyle\int_0^{1/2} \frac{1}{x^2} + \frac{1}{(x-1)^2}\, dx = \lim_{u\to 0^+}\left(-\frac{1}{x} - \frac{1}{x-1}\right)\Big]_u^{1/2}$

$$= \lim_{u\to 0^+}\left(\frac{1}{u} + \frac{1}{u-1}\right) = \infty.$$

Similarly we find that

$$\int_{1/2}^1 \frac{1}{x^2} + \frac{1}{(x-1)^2}\, dx = \infty.$$

In this situation we may write

$$\int_0^1 \frac{1}{x^2} + \frac{1}{(x-1)^2}\, dx = \infty$$

And we say that the region under the curve has infinite area.

◎ 函數為 $\dfrac{1}{x^2} + \dfrac{1}{(x-1)^2}$ ，區間範圍從 **0** 到 **-1**，可使用指令

『**integrate(函數, 變數, 範圍最小值, 範圍最大值);**』表示求積分

(%i74) integrate(1/x^2+1/(x-1)^2, x, 0, 1);

```
defint: integral is divergent.
 -- an error.  To debug this try debugmode(true);
```

(%i75) limit(1/x^2+1/(x-1)^2, x, 0, plus);

(%o75) ∞

Example 10   Find $\int_{-\infty}^{\infty} x \, dx$. We see that

$$\int_{-\infty}^{0} x \, dx = \lim_{u \to -\infty} \int_{u}^{0} x \, dx = \lim_{u \to -\infty} \frac{1}{2} x^2 \bigg]_{u}^{0} = -\infty,$$

and
$$\int_{0}^{\infty} x \, dx = \infty$$

Thus $\int_{-\infty}^{\infty} x \, dx$ diverges and has the form $\infty - \infty$. We do not assign it any value or either of the symbols ∞ to $-\infty$. The region under the curve is shown in Figure 6.7.10.

It is tempting to argue that the positive area to the right of the origin and the negative area to the left exactly cancel each other out so that the improper integral is zero. But this leads to a paradox.

Wrong: $\int_{-\infty}^{\infty} x \, dx = 0$.   Let $v = x + 2, \ dv = dx$.   Then

$$\int_{-\infty}^{\infty} (x+2) dx = \int_{-\infty}^{\infty} v \, dv = 0.$$

Subtracting $\int_{-\infty}^{\infty} (x+2) - x \, dx = 0 - 0 = 0$, $\int_{-\infty}^{\infty} 2 \, dx = 0$.

But $\int_{-\infty}^{\infty} 2 \, dx = \infty$.

So we do not give the integral $\int_{-\infty}^{\infty} x \, dx = \infty$ the value 0, and instead leave it undefined.

(%i81) integrate(x, x, minf, inf);

```
Principal Value
 (%o81) 0
```

(%i86) limit (integrate(x, x, minf, -h)+integrate(x, x, h,inf),h,0);

```
defint: integral is divergent.
 -- an error.  To debug this try debugmode(true);
```

(%i84) integrate(2, x, minf, inf);

```
defint: integral is divergent.
 -- an error.  To debug this try debugmode(true);
```

◎ 此題積分無法直接運算，奇異點在 0，必須跳開 0 這個點

分成 $-\infty$ 到 0^- 和 0^+ 到 ∞ 做積分再相加運算，再取極限 h 趨近於 0

有關 Principal Value

柯西主值 (Cauchy Principal Value)

因所欲解析之線積分是$-\infty$ 至$+\infty$ 的積分，且 $x = 0$ 為奇異點，故遇到不可解析

點 $x = 0$ 時需避開，只能假設從$-\infty$ 逼近一個 $-h$ 值 ，再從 h 值逼近$+\infty$

$$\lim_{h \to 0} \int_{-\infty}^{-h} x\,dx + \lim_{h \to 0} \int_{h}^{\infty} x\,dx = \int_{-\infty}^{\infty} x\,dx$$

求出結果為發散

http://people.chu.edu.tw/~cclu/ComplexVariableAnalysis/Summary%20400.pdf

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