# Maxima 在微積分上之應用

## 極限，幾何的分析與近似

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日期：2009/10/27

## 5.1 Infinite limits

Example 1.

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Solution：(%i1) limit(1/x^2,x,0); 極限指令：limit(方程式，極限變數，範圍) //
方程式為 $\frac{1}{x^2}$，極限變數為 x，範圍為 x 趨近於 0

(%o1)  ∞

Example 2.

$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Solution：(%i1) limit(1/x,x,0,plus); 極限指令：limit(方程式，極限變數，範圍) //
方程式為 $\frac{1}{x}$，極限變數為 x，範圍為 x 趨近於 0，plus 為取右極限

(%o1)  ∞

(%i2) limit(1/x,x,0,minus); 極限指令：limit(方程式，極限變數，範圍) //方程式
為 $\frac{1}{x}$，極限變數為 x，範圍為 x 趨近於 0，minus 為取左極限

(%o2)  −∞

Example 3.

Find $\lim_{x \to \infty} \frac{3x^4 + 5x - 2}{2x^4 - 6x^3 + 7}$.

Solution：(%i1) limit((3*x^4+5*x-2)/(2*x^4-6*x^3+7),x,inf); 極限指令：limit(方
程式，極限變數，範圍) //方程式為 $\frac{3x^4 + 5x - 2}{2x^4 - 6x^3 + 7}$，極限變數為 x，範圍為 x 趨
近於 ∞

(%o1)  $\frac{3}{2}$

Let $H$ be positive infinite. Then

$$\frac{3H^4 + 5H - 2}{2H^4 - 6H^3 + 7} = \frac{3 + 5H^{-3} - 2H^{-4}}{2 - 6H^{-1} + 7H^{-4}} \ ,$$

and therefore $st(\dfrac{3H^4 + 5H - 2}{2H^4 - 6H^3 + 7}) = \dfrac{3 + 0 - 0}{2 - 0 + 0} = \dfrac{3}{2}$.

Thus the limit exists and is $\dfrac{3}{2}$.


Example 4.

Find $\lim\limits_{x \to -\infty} (x^3 + 200x^2)$.

Solution：(%i1) limit(x^3+200*x^2,x,minf);　極限指令：limit(方程式，極限變數，範圍)　//方程式為 $x^3 + 200x^2$ ，極限變數為 x，範圍為 x 趨近於 $-\infty$

(%o1)　$-\infty$

We have $x^3 + 200x^2 = x^2(x + 200)$. When $H$ is negative infinite, $H^2$ is positive infinite and $(H + 200)$ is negative infinite, so their product is negative infinite.

Thus $\lim\limits_{x \to -\infty} (x^3 + 200x^2) = -\infty$.


Example 5.

A student can get a score of $100t/(t+1)$ on his math exam if he studies $t$ hours for it (Figure 5.1.2). If he studies infinitely long for the exam, his score will be infinitely close to 100, because if $H$ is positive infinite,

$$st(\frac{100H}{H+1}) = st(\frac{100}{1+1/H}) = \frac{100}{1+0} = 100 .$$

In the notation of limits,

$$\lim\limits_{t \to \infty} \frac{100t}{t+1} = 100.$$

(%i1) limit(100*t/(t+1),t,inf);　極限指令：limit(方程式，極限變數，範圍)　//方程式為 $\dfrac{100t}{t+1}$ ，極限變數為 t，範圍為 t 趨近於 $\infty$

(%o1)　100


Example 6.

Given any polynomial

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_1 t + a_0$$

Of degree $n > 0$, the limits as $t$ approaches $-\infty$ or $+\infty$ are as follows. Suppose $a_n > 0$. When $n$ is even, $\lim\limits_{t \to -\infty} f(t) = \infty$, $\lim\limits_{t \to \infty} f(t) = \infty$.

When $n$ is odd, $\lim\limits_{t \to -\infty} f(t) = -\infty$, $\lim\limits_{t \to \infty} f(t) = \infty$.

The signs are all reversed when $a_n < 0$.

All these limits can be computed from

$$f(t) = t^n (a_n + \frac{a_{n-1}}{t} + ... + \frac{a_1}{t^{n-1}} + \frac{a_0}{t^n}).$$

Example 7.

In the special theory of relativity, a body which is moving at constant velocity $v$, $-c < v < c$, will have mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

and its length in the direction of motion will be

$$l = l_0 \sqrt{1 - v^2/c^2} .$$

Here $m_0$, $l_0$, and $c$ are positive constants denoting the mass at rest (that is, the mass where $v = 0$), the length at rest, and the speed of light.

Suppose the velocity $v$ is infinitely close to the speed of light $c$, that is, $v = c - \varepsilon$, $\varepsilon > 0$ infinitesimal.

Then $\sqrt{1 - \dfrac{v^2}{c^2}} = \sqrt{1 - \dfrac{(c - \varepsilon)^2}{c^2}} = \sqrt{\dfrac{c^2 - (c^2 - 2c\varepsilon + \varepsilon^2)}{c^2}}$

$= \sqrt{\dfrac{2\varepsilon}{c} - \dfrac{\varepsilon^2}{c^2}} = \sqrt{\varepsilon.(\dfrac{2}{c} - \dfrac{\varepsilon}{c^2})} ,$

which is the square root of a positive infinitesimal. Thus $\sqrt{1 - v^2/c^2}$ is a positive infinitesimal. Therefore for $v$ infinitely close to $c$, $m$ is positive infinite and $l$ is positive infinitesimal. That is, a body moving at velocity infinitely close to (but less than ) the speed of light has infinite mass and infinitesimal length in the direction of motion. In the notation of limits this means that

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = +\infty ,$$

$$\lim_{v \to c^-} l_0 \sqrt{1 - v^2/c^2} = 0 .$$

Caution∶ This example must be understood in the light of our policy of speaking as if a line in physical space really is like the hyperreal line. Actually, there is no evidence

one way or the other on whether a line in space is like the hyperreal line, but the hyperreal line is a useful model for the purpose of applications.

Example 8.

Evaluate $\lim_{x\to\infty}\dfrac{\sin x}{x}$ .

Solution：(%i1) limit(sin(x)/x,x,inf);　極限指令：limit(方程式，極限變數，範圍)

//方程式為 $\dfrac{\sin x}{x}$ ，極限變數為 x，範圍為 x 趨近於 $\infty$

   (%o1)　0

When $H$ is positive infinite, $\sin H$ is between -1 and 1 and thus finite, so $(\sin H)/H$ is infinitesimal. The limit is therefore zero：

$\lim_{x\to\infty}\dfrac{\sin x}{x}=0$ .

Example 9.

Find $\lim_{x\to\infty}\cos x$ .

Solution：(%i1) limit(cos(x),x,inf);　極限指令：limit(方程式，極限變數，範圍)　//
方程式為 $\cos x$ ，極限變數為 x，範圍為 x 趨近於 $\infty$ ，此題輸出的答案為 ind，ind represents a bounded, indefinite result

   (%o1)　ind

If $H$ is any integer or hyperinteger, then
$\cos(2\pi H)=1,\quad \cos(2\pi H+\pi)=-1$ .

In fact, $\cos x$ will keep oscillating between 1 and -1 even for infinite $x$ .
Therefore the limit does not exist.

## 5.2 L'Hospital's rule

Example 1.

Find $\lim_{x\to 1}\dfrac{(1/x)-1}{\sqrt{x}-1}$ .

Solution：(%i1) limit(((((1/x)-1)/(sqrt(x)-1)),x,1);　極限指令：limit(方程式，極限變

數，範圍)　//此例 f 為前所定義之方程式 $\dfrac{(1/x)-1}{\sqrt{x}-1}$ ，極限變數為 x，範圍為 x 趨

(%o1) −2

Both $(1/x) - 1$ and $\sqrt{x} - 1$ approach 0 as $x$ approaches 1. The limit is thus of the form $0/0$. Using l'Hospital's Rule,

$$\lim_{x \to 1} \frac{(1/x) - 1}{\sqrt{x} - 1} = \lim_{x \to 1} \frac{-x^{-2}}{\frac{1}{2} x^{-1/2}} = \frac{-1}{\frac{1}{2}} = -2.$$

Example 2.

Find $\lim_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x^3}$ .

Solution：(%i1) limit((sqrt(x+1)-1)/(x^3),x,0); 極限指令：limit(方程式，極限變數，範圍) //方程式爲 $\dfrac{\sqrt{x+1}-1}{x^3}$ ，極限變數爲 x，範圍爲 x 趨近於 0

(%o1) ∞

The limit is of the form 0/0. The limit of $f'(x)/g'(x)$ as $x \to 0$ is $\infty$,

$$\lim_{x \to 0} \frac{d(\sqrt{x+1} - 1)/dx}{d(x^3)/dx} = \lim_{x \to 0} \frac{\frac{1}{2}(x+1)^{-1/2}}{3x^2} = \infty .$$

Thus by l'Hospital's Rule,

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x^3} = \infty .$$

Example 3.

Find $\lim_{x \to 3}(x + \dfrac{1}{x-3})(\sqrt{x+1} - 2).$

Solution：(%i1) limit((x+1/(x-3))*(sqrt(x+1)-2),x,3); 極限指令：limit(方程式，極限變數，範圍) //方程式爲 $(x + \dfrac{1}{x-3})(\sqrt{x+1} - 2)$ ，極限變數爲 x，範圍爲 x 趨近於 3

(%o1) $\dfrac{1}{4}$

The limit is not in a form where we can apply l'Hospital's Rule. We must first use algebra to put it in another form,

$$(x + \frac{1}{x-3})(\sqrt{x+1} - 2) = x(\sqrt{x+1} - 2) + \frac{\sqrt{x+1} - 2}{x - 3}.$$

By elementary computations, $\lim\limits_{x \to 3} x(\sqrt{x+1} - 2) = 3.0 = 0.$

Using l'Hospital's Rule,

$$\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3} = \lim_{x \to 3} \frac{\frac{1}{2}(x+1)^{-1/2}}{1} = \frac{1}{2}.4^{-1/2} = \frac{1}{4}.$$

We then add the limits to get the desired answer,

$$\lim_{x \to 3}(x + \frac{1}{x-3})(\sqrt{x+1} - 2) = \lim_{x \to 3} x(\sqrt{x+1} - 2) + \lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3}$$
$$= 0 + \frac{1}{4} = \frac{1}{4}.$$

Example 4.

Find $\lim\limits_{x \to 1} \dfrac{\dfrac{(x-3)}{4} + \dfrac{1}{x+1}}{(x+1)^2}.$

Solution：(%i1) limit(((x-3)/4+1/(x+1))/(x-1)^2,x,1);　　極限指令：limit(方程式，

極限變數，範圍)　//方程式為 $\dfrac{\dfrac{(x-3)}{4} + \dfrac{1}{x+1}}{(x+1)^2}$ ，極限變數為 x，範圍為 x 趨近於 1

(%o1)　$\dfrac{1}{8}$

This limit is of the form 0/0. When l'Hospital's Rule is used the limit is still of the form 0/0. But when it is used a second time we can compute the limit.

$$\lim_{x \to 1} \frac{\frac{(x-3)}{4} + \frac{1}{x+1}}{(x-1)^2} = \lim_{x \to 1} \frac{\frac{1}{4} - \frac{1}{(x+1)^2}}{2(x-1)} = \lim_{x \to 1} \frac{2(x+1)^{-3}}{2} = \frac{1}{8}.$$

Example 5.

Find $\lim\limits_{x \to 0^+} \dfrac{\sqrt{x+4} - 2}{\sqrt{x}}.$

Solution：(%i1) limit((sqrt(x+4)-2)/sqrt(x),x,0,plus);　極限指令：limit(方程式，極限變數，範圍)　//方程式為 $\dfrac{\sqrt{x+4}-2}{\sqrt{x}}$ ，極限變數為 x，範圍為 x 趨近於 0，plus 為取右極限

(%o1)　0

The limit as $x \to 0$ does not exist because $\sqrt{x}$ is defined only for $x > 0$. However, the one-sided limit as $x \to 0^+$ has the form 0/0 and can be found by l'Hospital Rule.

$$\lim_{x \to 0^+} \frac{\sqrt{x+4}-2}{\sqrt{x}} = \lim_{x \to 0^+} \frac{\frac{1}{2}(x+4)^{-1/2}}{\frac{1}{2}x^{-1/2}} = \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{x+4}} = 0.$$

Example 6.

Find $\lim_{x \to \infty} \dfrac{x + \sqrt{x} + 1}{\sqrt{x} + \sqrt{x+1}}$.

Solution：(%i1) limit((x+sqrt(x)+1)/(sqrt(x)+sqrt(x+1)),x,inf);　極限指令：limit(方程式，極限變數，範圍)　//方程式為 $\dfrac{x + \sqrt{x} + 1}{\sqrt{x} + \sqrt{x+1}}$ ，極限變數為 x，範圍為 x 趨近於 ∞

(%o1)　∞

By l'Hospital's Rule for $\infty / \infty$,

$$\lim_{x \to \infty} \frac{x + \sqrt{x} + 1}{\sqrt{x} + \sqrt{x+1}} = \lim_{x \to \infty} \frac{1 + \dfrac{1}{2\sqrt{x}}}{\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{x+1}}} = \infty.$$

Example 7.

Find $\lim_{x \to 1} \dfrac{\sqrt{x} - (1/x)}{x}$.

Solution：(%i1) limit((sqrt(x)-(1/x))/x,x,1);　極限指令：limit(方程式，極限變數，範圍)　//方程式為 $\dfrac{\sqrt{x}-(1/x)}{x}$ ，極限變數為 x，範圍為 x 趨近於 1

(%o1)　0

The limit has the form 0/1, so l'Hospital's Rule does not apply.

*Correct*：$\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-(1/x)}{x}=\frac{\displaystyle\lim_{x\to 1}(\sqrt{x}-(1/x))}{\displaystyle\lim_{x\to 1}x}=\frac{0}{1}=0.$

*Incorrect*：$\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-(1/x)}{x}=\lim_{x\to 1}\frac{d(\sqrt{x}-(1/x))/dx}{dx/dx}=\lim_{x\to 1}(\frac{1}{2\sqrt{x}}+\frac{1}{x^2})=\frac{3}{2}.$

## 5.3 Limits and Curve Sketching

Example 1.

Consider the limit $\displaystyle\lim_{x\to 0}\frac{2/x+1}{1/x-1}=2.$

Solution：(%i1) limit((2/x+1)/(1/x+1),x,0);　極限指令：limit(方程式，極限變數，範圍)　//方程式為 $\dfrac{2/x+1}{1/x-1}$ ，極限變數為 x，範圍為 x 趨近於 0

(%o1)　2

This limit is evaluated by letting $x \neq 0$ be infinitesimal：

$\dfrac{2/x+1}{1/x-1}=\dfrac{2+x}{1-x},$

$\displaystyle\lim_{x\to 0}\frac{2/x+1}{1/x-1}=st(\frac{2+x}{1-x})=\frac{st(2+x)}{st(1-x)}=\frac{2+0}{1-0}=2$ .

Let us see what happens if instead of taking $x$ to be infinitely small we take $x$ to be a "small" real number. We shall make a table of values of $f(x)=\dfrac{2/x+1}{1/x-1}$ for various small $x$ .

| $x$ | $f(x)=\dfrac{2/x+1}{1/x-1}$ | $f(x)$ to four places |
|---|---|---|
| 0.1 | 21/9 | 2.3333 |
| 0001 | 201/99 | 2.0303 |
| 0.001 | 2001/999 | 2.0030 |

| 0.0001 | 20001/9999 | 2.0003 |
|--------|------------|--------|
| -0.1 | 19/11 | 1.7364 |
| -0.01 | 199/101 | 1.9703 |
| -0.001 | 1999/1001 | 1.9970 |
| -0.0001 | 19999/10001 | 1.9997 |

We see that as $x$ gets closer and closer to zero, $f(x)$ gets closer and closer to 2.

Example 2.

Consider the limit $\lim\limits_{x \to 2} \dfrac{1}{(x-2)^2} = \infty$.

Solution：(%i1) limit(1/(x-2)^2,x,2); 極限指令：limit(方程式，極限變數，範圍)

//方程式為 $\dfrac{1}{(x-2)^2}$ ，極限變數為 x，範圍為 x 趨近於 2

(%o1)  ∞

For $x$ infinitely close but not equal to 2, $1/(x-2)^2$ is positive infinite. Let us make a table of values when $x$ is a real number close to but not equal to 2.

| $x$ | $f(x)$ |
|-----|--------|
| 2.1 | 100 |
| 2.01 | 10000 |
| 2.001 | 1000000 |
| 1.9 | 100 |
| 1.99 | 10000 |
| 1.999 | 1000000 |

As $x$ gets closer and closer to 2, $f(x)$ gets larger and larger.

Example 3.

$\lim\limits_{x \to \infty}(1 + \dfrac{1}{(x-2)^2}) = 1$.

Solution：(%i1) limit(1+1/(x-2)^2,x,inf); 極限指令：limit(方程式，極限變數，範圍)  //方程式為 $1 + \dfrac{1}{(x-2)^2}$ ，極限變數為 x，範圍為 x 趨近於 ∞

(%o1)  1

For infinitely large $x$, $1+1/(x-2)^2$ is infinitely close to 1. Here is a table of values of $1+1/(x-2)^2$ for large real $x$.

| $x$ | $1+\dfrac{1}{(x-2)^2}$ |
|---|---|
| 12 | 1.01 |
| 102 | 1.0001 |
| 1002 | 1.000001 |
| 10002 | 1.00000001 |

As $x$ gets large, $1+1/(x-2)^2$ gets close to 1. Also notice that

$$\lim_{x\to-\infty}(1+\frac{1}{(x-2)^2})=1,$$

and for large negative $x$, $1+1/(x-2)^2$ is close to 1.

Example 4.

Sketch the curve $f(x)=1+\dfrac{1}{(x-2)^2}$.

Solution：(%i1) plot2d(1+1/(x-2)^2,[x,-10,10],[y,0,10]);　繪圖指令解說：
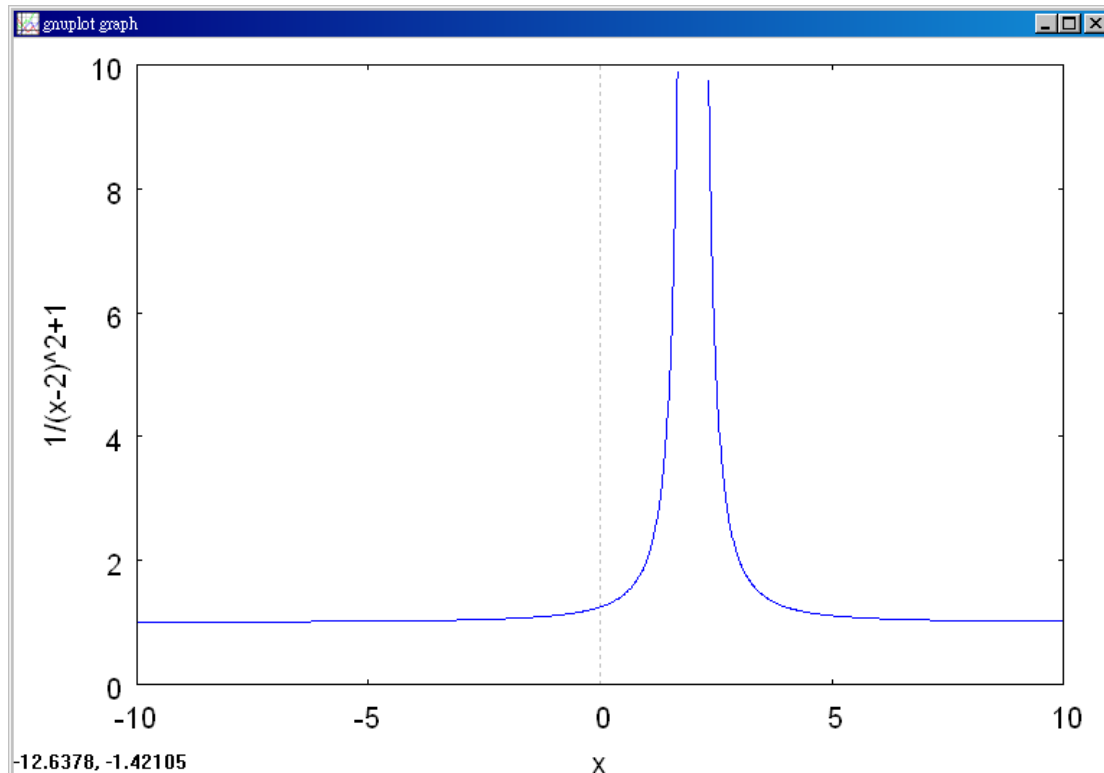plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $1+\dfrac{1}{(x-2)^2}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍 0~10，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型····等等。

```
plot2d: some values were clipped.
  (%o1)
```

(%i2) diff(1+1/(x-2)^2,x);    微分的指令：differ(函數，要微分的變數)    //對

$1 + \dfrac{1}{(x-2)^2}$ 中的 x 變數微分

(%o2)    $-\dfrac{2}{(x-2)^3}$

(%i3) diff(1+1/(x-2)^2,x,2);    微分的指令：differ(函數，要微分的變數，次數)    //

對 $1 + \dfrac{1}{(x-2)^2}$ 中的 x 變數微分 2 次

(%o3)    $\dfrac{6}{(x-2)^4}$

The first two derivatives are

$f'(x) = -2(x-2)^{-3}$                $f''(x) = 6(x-2)^{-4}$.

The first and second derivatives are never zero. $f(x)$ is undefined at $x = 2$. In our table we shall show the values of $f(x)$ and its first two derivatives at a point on each side of $x = 2$. We shall also show the limits of $f(x)$ and its first derivative as $x \to -\infty$, $x \to 2^-$, $x \to 2^+$, and $x \to \infty$. (We will not need the limits of $f''(x)$.)

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Comments |
|---|---|---|---|---|
| $\lim_{x\to-\infty}$ | 1 | 0 | | Horizontal |
| $x=1$ | 2 | 2 | 6 | Increasing, $\cup$ |
| $\lim_{x\to2^-}$ | $\infty$ | $\infty$ | | Vertical |
| $\lim_{x\to2^+}$ | $\infty$ | $-\infty$ | | Vertical |
| $x=3$ | 2 | -2 | 6 | Decreasing, $\cup$ |
| $\lim_{x\to\infty}$ | 1 | 0 | | horizontal |

The first line of the table, $\lim_{x\to-\infty}$ , shows that for large negative $x$ the curve is close to

1 and its slope is nearly horizontal. The second line, $x=1$, shows that the curve is increasing and concave upward in the interval $(-\infty,2)$, and passes through the point

(1, 2) with a slope of 2. The third line, $\lim_{x\to2^-}$ , shows that just before $x=2$ the curve

is far above the $x-$ axis and its slope is nearly vertical. Going through the table in this way, we are able to sketch the curve as in Figure 5.3.2.

The curve approaches the dotted horizontal line $y=1$ and the dotted vertical line $x=2$. The lines are called asymptotes of the curve,.

Example 5.
$f(x) = x^{3/5}$.

Solution：(%i1) plot2d(x^(3/5),[x,-2,2],[y,-2,2]);　繪圖指令解說：plot2d([expr，x_range，options])，　plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

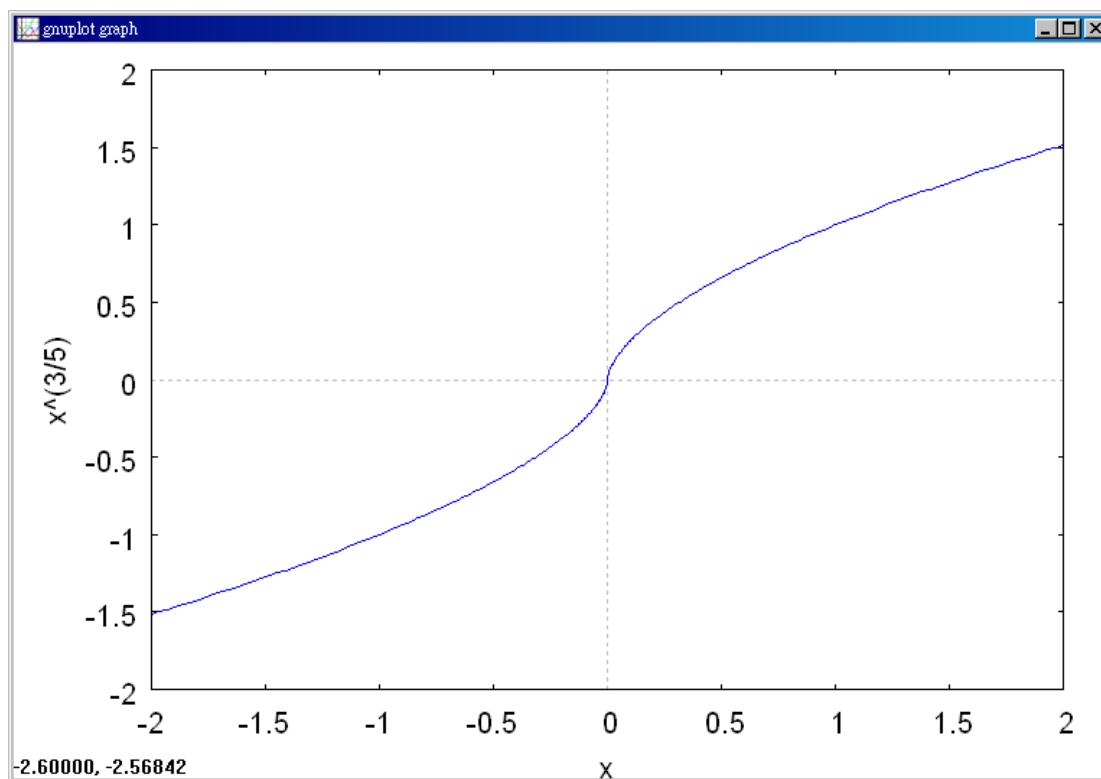expr：是你要繪製的函數，這例是 $x^{3/5}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-2~2，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等

等。

(%i2) diff(x^(3/5),x);  微分的指令：differ(函數，要微分的變數)  //對 $x^{3/5}$ 中的 x 變數微分

$$(\%o2) \quad \frac{3}{5\,x^{2/5}}$$

(%i3) diff(x^(3/5),x,2);  微分的指令：differ(函數，要微分的變數，次數)  //對 $x^{3/5}$ 中的 x 變數微分 2 次

$$(\%o3) \quad -\frac{6}{25\,x^{7/5}}$$

Then $f'(x) = \dfrac{3}{5}x^{-2/5}$, $f''(x) = -\dfrac{6}{25}x^{-7/5}$.

At the point $x = 0$, $f(x) = 0$ and $f'(x)$ does not exist. We first plot a few points, compute the necessary limits, and make a table.

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Comments |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| $\lim\limits_{x\to-\infty}$ | $-\infty$ | 0 | | Horizontal |
| $x=-1$ | -1 | 3/5 | 6/25 | Increasing, $\cup$ |
| $\lim\limits_{x\to0^-}$ | 0 | $\infty$ | | Vertical |
| $x=0$ | 0 | Undef. | | |
| $\lim\limits_{x\to0^+}$ | 0 | $\infty$ | | Vertical |
| $x=1$ | 1 | 3/5 | -6/25 | Increasing, $\cup$ |
| $\lim\limits_{x\to\infty}$ | $\infty$ | 0 | | Horizontal |

Example 6.

$f(x)=x^{4/5}$.

Solution：(%i1) plot2d(x^(4/5),[x,-2,2],[y,-2,2]);　繪圖指令解說：plot2d([expr，x_range，options])，　plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $x^{4/5}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-2~2，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

(%o1)
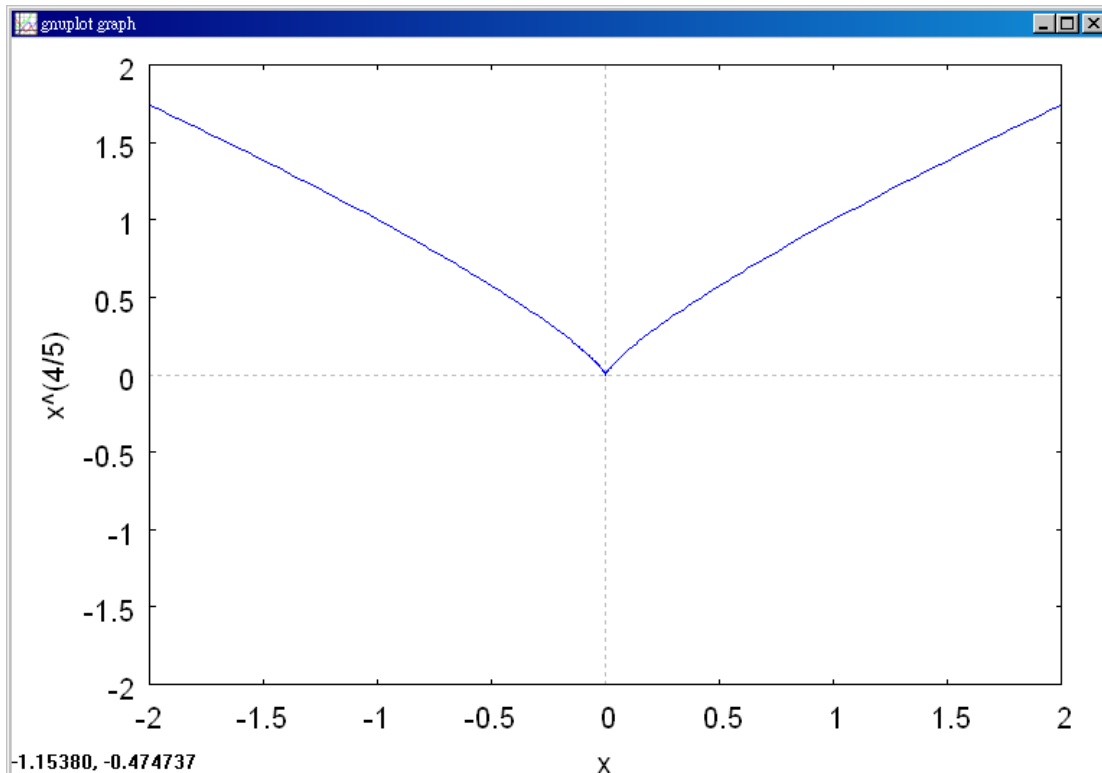
(%i2) diff(x^(4/5),x);   微分的指令：differ(函數，要微分的變數)   //對 $x^{4/5}$ 中的 x 變數微分

(%o2)   $\dfrac{4}{5\,x^{1/5}}$

(%i3) diff(x^(4/5),x,2);   微分的指令：differ(函數，要微分的變數，次數)   //對 $x^{4/5}$ 中的 x 變數微分 2 次

(%o3)   $-\dfrac{4}{25\,x^{6/5}}$

Then   $f'(x) = \dfrac{4}{5}x^{-1/5}$,   $f''(x) = -\dfrac{4}{25}x^{-6/5}$.

$f'(x)$   is undefined at   $x = 0$. We make the table：

| | $f(x)$ | $f'(x)$ | $f''(x)$ | Comments |
|---|---|---|---|---|
| $\lim\limits_{x \to -\infty}$ | $\infty$ | 0 | | Horizontal |
| $x = -1$ | 1 | -4/5 | -4/25 | Decreasing, $\cap$ |

| | | | | |
|---|---|---|---|---|
| $\lim\limits_{x\to 0^-}$ | 0 | $-\infty$ | | Vertical |
| $x = 0$ | 0 | Undef. | | |
| $\lim\limits_{x\to 0^+}$ | 0 | $\infty$ | | Vertical |
| $x = 1$ | 1 | 4/5 | -4/25 | Increasing, $\cap$ |
| $\lim\limits_{x\to\infty}$ | $\infty$ | 0 | | Horizontal |

Example 7.

Sketch the curve $f(x) = \dfrac{\cos x}{\sin x}$ for $0 < x < 2\pi$.

Solution：(%i1) plot2d(cos(x)/sin(x),[x,0,2*%pi],[y,-3,3]); 繪圖指令解說：
plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。
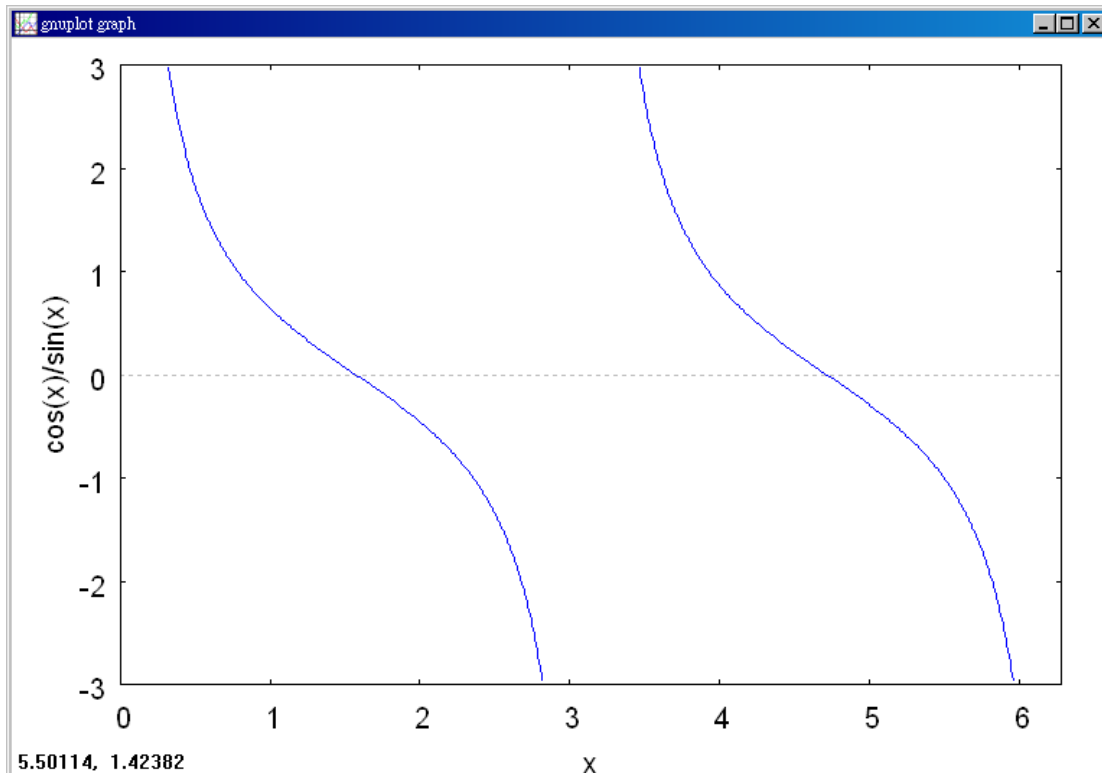
expr：是你要繪製的函數，這例是 $\dfrac{\cos x}{\sin x}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-3~3，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‥‥等等。

```
plot2d: expression evaluates to non-numeric value somewhere in plot
plot2d: some values were clipped.
  (%o1)
```

(%i2) diff(cos(x)/sin(x),x); 微分的指令：differ(函數，要微分的變數) //對 $\dfrac{\cos x}{\sin x}$

中的 x 變數微分

(%o2)   $-\dfrac{\cos(x)^2}{\sin(x)^2}-1$

(%i3) diff(cos(x)/sin(x),x,2); 微分的指令：differ(函數，要微分的變數，次數) //

對 $\dfrac{\cos x}{\sin x}$ 中的 x 變數微分 2 次

(%o3)   $\dfrac{2\cos(x)}{\sin(x)}+\dfrac{2\cos(x)^3}{\sin(x)^3}$

$f(x)$ and $f'(x)$ are undefined at $x=\pi$ because the denominator $\sin\pi$ is zero.

The first two derivatives are

$$f'(x)=-\dfrac{1}{\sin^2 x}, \quad f''(x)=2\dfrac{\cos x}{\sin^3 x}.$$

Thus $f'(x)$ is always negative, and $f''(x) = 0$ when $x = \pi/2, 3\pi/2$. Here is the table：

|  | $f(x)$ | $f'(x)$ | $f''(x)$ | Comments |
|---|---|---|---|---|
| $\lim\limits_{x \to 0^+}$ | $\infty$ | $-\infty$ |  | Vertical |
| $\pi/4$ | 1 | -1/2 | + | Decreasing, $\cup$ |
| $\pi/2$ | 0 | -1 | 0 | Decreasing, inflection |
| $3\pi/4$ | -1 | -1/2 | — | Decreasing, $\cap$ |
| $\lim\limits_{x \to \pi^-}$ | $-\infty$ | $-\infty$ |  | Vertical |
| $\lim\limits_{x \to \pi^+}$ | $\infty$ | $-\infty$ |  | Vertical |
| $5\pi/4$ | 1 | -1/2 | + | Decreasing, $\cup$ |
| $3\pi/2$ | 0 | -1 | 0 | Decreasing, inflection |
| $7\pi/4$ | -1 | -1/2 | — | Decreasing, $\cap$ |
| $\lim\limits_{x \to 2\pi^-}$ | $-\infty$ | $-\infty$ |  | Vertical |

Notice that the table from $\pi$ to $2\pi$ is just a repeat of the table from 0 to $\pi$. This because $\dfrac{\cos(x + \pi)}{\sin(x + \pi)} = \dfrac{-\cos x}{-\sin x} = \dfrac{\cos x}{\sin x}$.

The curve is sketched in Figure 5.3.5.

## 5.4 Parabolas

Example 1.

Find an equation for the vertical parabola with directrix $y = -1$ and focus $F(0,1)$ (Figure 5.4.6).

Solution：

Given a point $P(x, y)$, the perpendicular from $P$ to the directrix is a vertical line of length $\sqrt{(y + 1)^2}$. Thus

distance from $P$ to directrix $= \sqrt{(y+1)^2}$ .

Also, distance from $P$ to focus $= \sqrt{x^2 + (y-1)^2}$ .

The point $P$ lies on the parabola exactly when these distances are equal,

$$\sqrt{(y+1)^2} = \sqrt{x^2 + (y-1)^2} .$$

**Example 2.**

Find the focus and directrix of the parabola $y = -2x^2$.

Solution：In Theorem 1, $a = -2$ and $d = \dfrac{1}{4}a = -\dfrac{1}{2}$. The focus is $F(0, -\dfrac{1}{2})$, and the directrix is $y = \dfrac{1}{2}$.

**Example 3.**

Find the vertex, focus and directrix of the parabola $y = 2x^2 - 5x + 4$.

Solution：First find the point $x_0$ where the slope is 0.

$$\frac{dy}{dx} = 4x - 5 .$$

Then $4x_0 - 5 = 0$,

$$x_0 = \frac{5}{4} .$$

Substitute to find $y_0$.

$$y_0 = 2(x_0)^2 - 5x_0 + 4 = \frac{7}{8} .$$

The vertex is $(x_0, y_0) = (\frac{5}{4}, \frac{7}{8})$.

We have $a = 2$, so $\frac{1}{4}a = \frac{1}{8}$. By Theorem 2, the focus is

$$(x_0, y_0 + \frac{1}{4a}) = (\frac{5}{4}, 1).$$

The directrix is $y = y_0 - \frac{1}{4a}$, $y = \frac{3}{4}$.

The vertex, axis, focus, and directrix can be used to sketch quickly the graph of a parabola.

Example 2. (Continued)

Sketch the parabola $y = -\frac{1}{2}x^2$.

Solution：(%i1) plot2d(-(1/2)*x^2,[x,-10,10],[y,-10,10]);　　繪圖指令解說：
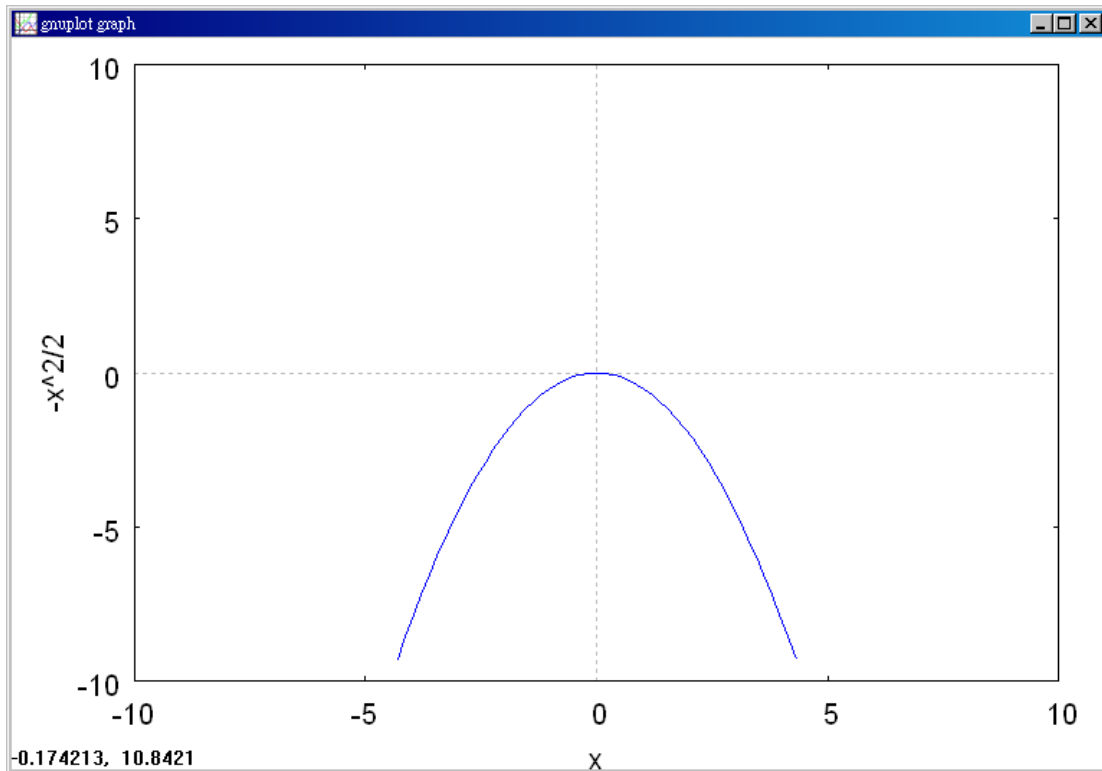plot2d([expr，x_range，options])，　plot2d 是 Maxima 的繪圖指令，maxima 執行
到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $y = -\frac{1}{2}x^2$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y
軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指
定 x 軸，這裡我們也指定了 y 軸的範圍-10~10，另外函數中的變數要與範圍指定
的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‥‥等
等。

```
plot2d: some values were clipped.
  (%o1)
```

The first two derivatives are

$$\frac{dy}{dx} = -x, \quad \frac{d^2 y}{dx^2} = -1.$$

The only critical point is at $x = 0$. The table of values follows.

| $x$ | $y$ | $dy/dx$ | $d^2 y/dx^2$ | Comments |
|---|---|---|---|---|
| $\lim\limits_{x \to -\infty}$ | $-\infty$ | $\infty$ | | Vertical |
| $x = 0$ | $0$ | $0$ | -1 | Max, $\cap$ |
| $\lim\limits_{x \to \infty}$ | $-\infty$ | $-\infty$ | | vertical |

The parabola is drawn in Figure 5.4.9, using Steps 1-5.

Example 3. (Continued)

Sketch the parabola $y = 2x^2 - 5x + 4$.

Solution：

The first two derivatives are

$$\frac{dy}{dx} = 4x - 5, \quad \frac{d^2 y}{dx^2} = 4.$$

The only critical point is at the vertex, where $x = \dfrac{5}{4}$. The table of values follows.

| $x$ | $y$ | $dy/dx$ | $d^2 y/dx^2$ | Comments |
|---|---|---|---|---|
| $\lim\limits_{x \to -\infty}$ | $\infty$ | $-\infty$ | | Vertical |
| $5/4$ | $7/8$ | $0$ | $+$ | Min, $\cup$ |
| $\lim\limits_{x \to \infty}$ | $\infty$ | $\infty$ | | Vertical |

The parabola is drawn in Figure 5.4.10, again using Steps 1-5.

## 5.5　Ellipses and Hyperbolas

Example 1.

Sketch the curve $\dfrac{x^2}{9} + y^2 = 1$.

Solution：(%i1) load(implicit_plot);　Maxima 本身不會畫出隱函數的圖，需要讀取模組中的指令來執行，而隱函數的畫圖須讀取模組"implicit_plot" //讀取隱函數畫圖的模組"implicit_plot"

(%o1)
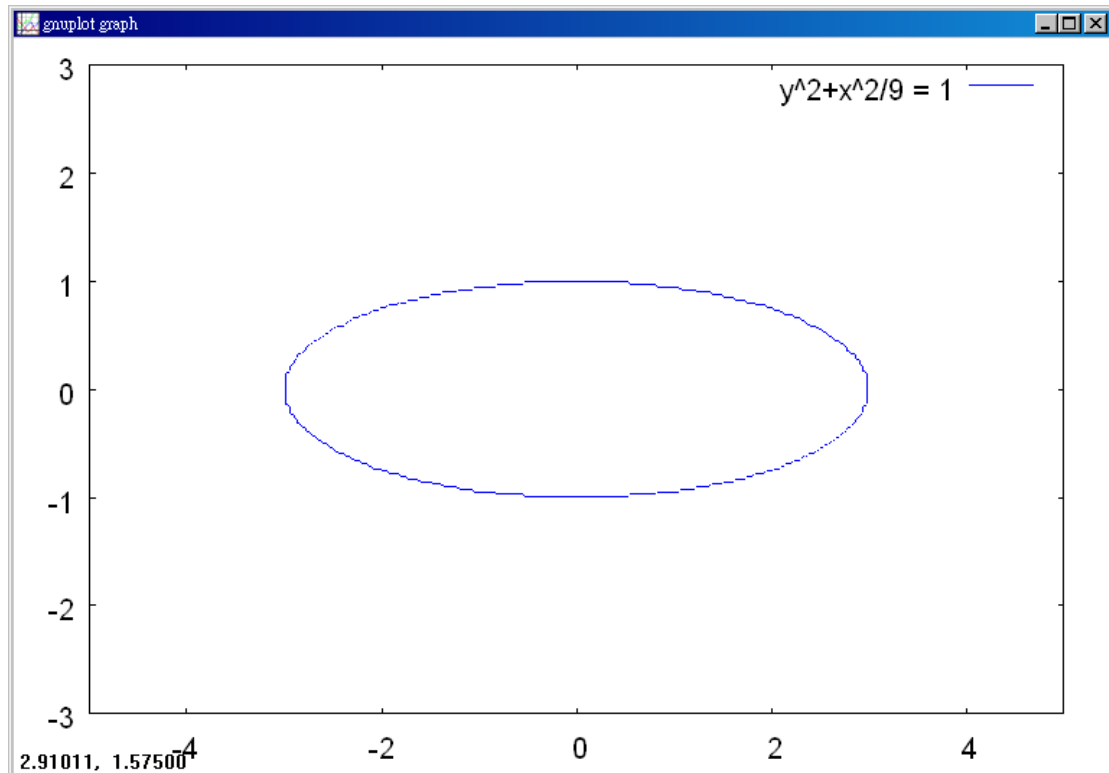C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/implicit_p.

(%i2) implicit_plot ((x^2)/9+y^2=1,[x,-5,5],[y,-3,3]);　隱函數畫圖指令：implicit_plot(方程式，x 軸範圍，y 軸範圍)

(%o2)　done

The curve is an ellipse that cuts the $x$-axis at $\pm 3$ and the $y$-axis at $\pm 1$. To sketch the curve, we first draw the rectangle $x = \pm 3$, $y = \pm 1$ with dotted lines and then inscribe the ellipse in the rectangle. The ellipse, shown in Figure 5.5.4, is horizontal.

Example 2.

Sketch the curve $4x^2 + y^2 = 9$ and find the foci.

Solution：(%i1) load(implicit_plot);　Maxima 本身不會畫出隱函數的圖，需要讀取模組中的指令來執行，而隱函數的畫圖須讀取模組"implicit_plot" //讀取隱函數畫圖的模組"implicit_plot"
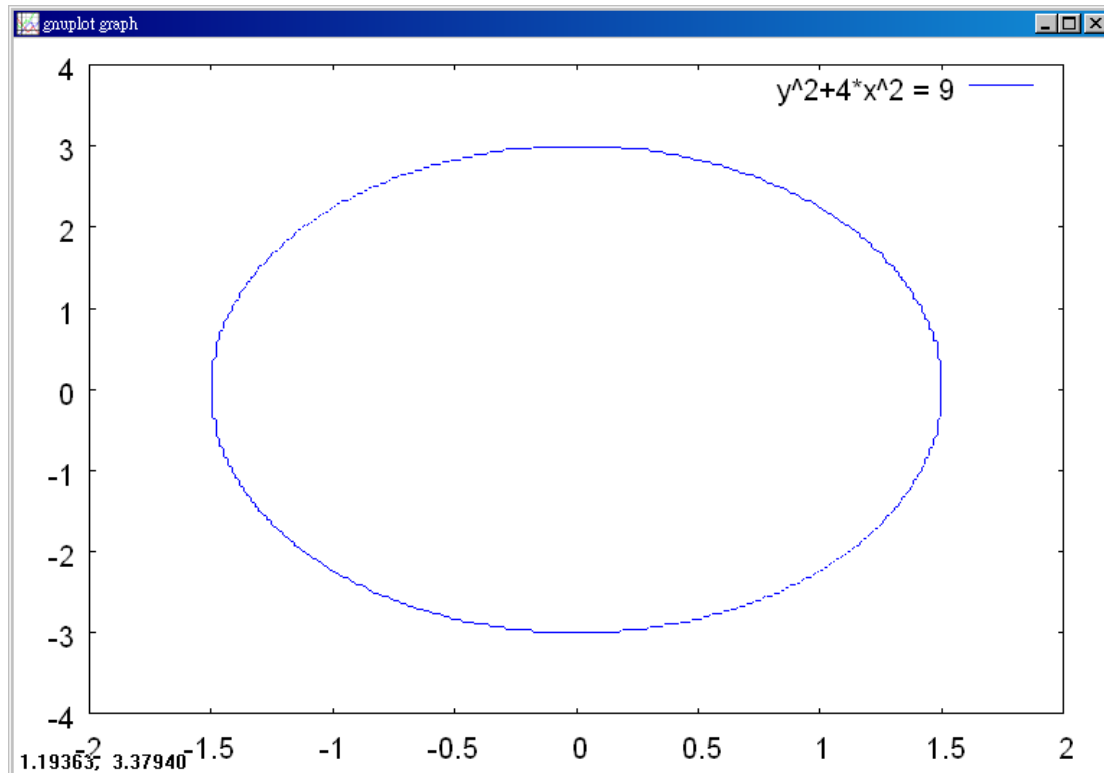
```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/implicit_p.
```

(%i2) implicit_plot (4*x^2+y^2=9,[x,-2,2],[y,-4,4]);　隱函數畫圖指令：implicit_plot(方程式，x 軸範圍，y 軸範圍)

```
(%o2)  done
```

The equation may be rewritten as

$$\frac{4}{9}x^2 + \frac{1}{9}y^2 = 1.$$

The graph (Figure 5.5.5) is a vertical ellipse cutting the $x$-axis at $\pm\dfrac{3}{2}$ and the $y$-axis at $\pm 3$.

By Theorem 1, the foci are on the $y$-axis at $(0,\ \pm c)$. We compute $c$ from the equation

$$c^2 = b^2 - a^2.$$

$a$ and $b$ are the $x$ and $y$ intercepts of the ellipse, $a = \dfrac{3}{2}$, $b = 3$. Thus

$$c^2 = 3^2 - (\frac{3}{2})^2 = \frac{27}{4}$$

$$c = \sqrt{\frac{27}{2}} \sim 2.598.$$

The foci are at $(0, \ \pm 2.598)$.

Example 3.

Sketch the hyperbola $4y^2 - x^2 = 1$ and find its foci.

Solution：(%i1) load(implicit_plot);　Maxima 本身不會畫出隱函數的圖，需要讀取模組中的指令來執行，而隱函數的畫圖須讀取模組"implicit_plot" //讀取隱函數畫圖的模組"implicit_plot"
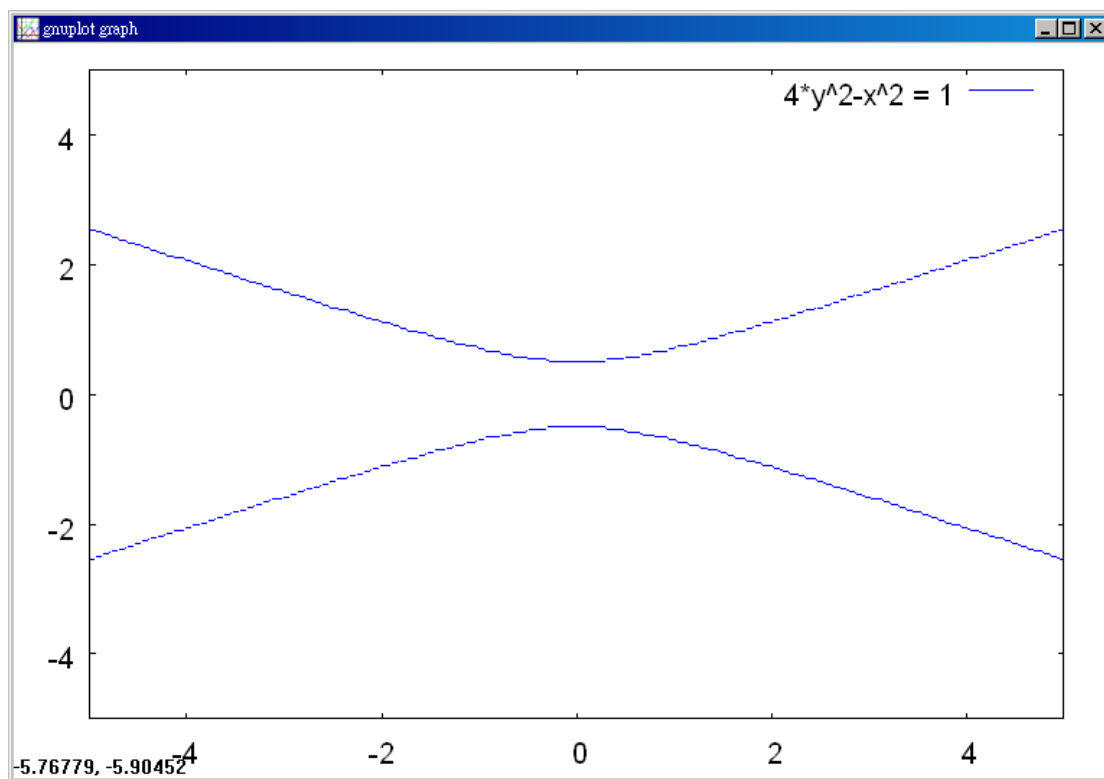
```
 (%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/implicit_p.
```

(%i2) implicit_plot (4*y^2-x^2=1,[x,-5,5],[y,-5,5]);　隱函數畫圖指令：implicit_plot(方程式，x 軸範圍，y 軸範圍)

```
 (%o2)  done
```



First compute $a$ and $b$.

$$4y^2 = y^2/b^2, \quad b = \frac{1}{2}$$

$$x^2 = x^2/a^2, \quad a = 1.$$

The rectangle has sides $x = \pm 1$, $y = \pm\frac{1}{2}$, and the vertices are at $(0, \pm\frac{1}{2})$. The hyperbola is sketched using Steps 1-4 in Figure 5.5.8. The foci are at $(0, \pm c)$ where

$$c^2 = a^2 + b^2 = 1^2 + (\frac{1}{2})^2 = 1.25$$

$$c = \sqrt{1.25} \approx 1.118.$$

## 5.6　Second Degree Curves

Example 1.

Sketch the curve $4x^2 - y^2 - 16x - 2y + 11 = 0$.

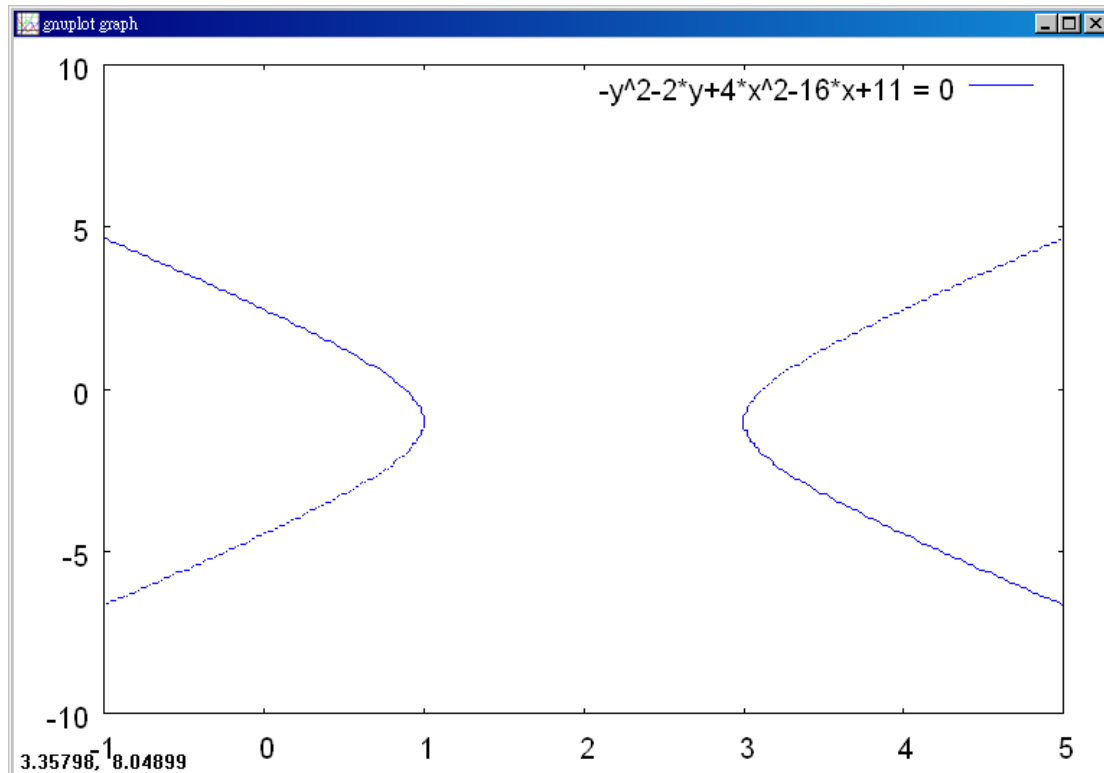Solution：(%i1) load(implicit_plot);　Maxima 本身不會畫出隱函數的圖，需要讀取模組中的指令來執行，而隱函數的畫圖須讀取模組"implicit_plot" //讀取隱函數畫圖的模組"implicit_plot"

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/contrib/implicit_p.

(%i2) implicit_plot (4*x^2-y^2-16*x-2*y+11=0,[x,-1,5],[y,-10,10]);　隱函數畫圖指令：implicit_plot(方程式，x 軸範圍，y 軸範圍)

(%o2)　done

Graph window titled "gnuplot graph" showing the curve -y^2-2*y+4*x^2-16*x+11 = 0, with axes from -10 to 10 (vertical) and from 0 to 5 (horizontal). Coordinate label: 3.35798, -8.04899

*Step1* Apply the Discriminant Test to determine the type of curve.

$$B^2 - 4AC = 0^2 - 4 \cdot 4 \cdot (-1) = 16.$$

The discriminant is positive, so the graph is a hyperbola.

*Step 2* Simplify by completing the squares. This is done by putting

$$X = x + \frac{D}{2A}, \quad Y = y + \frac{E}{2C}$$

and writing the original equation in terms of $X$ and $Y$.

$$X = x + \frac{-16}{2 \cdot 4} = x - 2, \quad x = X + 2$$

$$Y = y + \frac{-2}{2 \cdot (-1)} = y + 1, \quad y = Y - 1$$

$$4(X + 2)^2 - (Y - 1)^2 - 16(X + 2) - 2(Y - 1) + 11 = 0$$

$$4(X^2 + 4X + 4) - 16(X + 2) - (Y^2 - 2Y + 1) - 2(Y - 1) + 11 = 0.$$

The $X$ and $Y$ terms cancel, and

$$4X^2 + 16 - 32 - Y^2 - 1 + 2 + 11 = 0,$$

$$4X^2 - Y^2 - 4 = 0.$$

*Step 3* Draw dotted lines for the $X$ and $Y$ axes, and sketch the curve as in Section 5.5. This is a hyperbola in the $(X,Y)$-plane. The $X$-axis is the line $Y = 0$, or $y = -1$. The $Y$-axis is the line $X = 0$, or $x = 2$. The graph is shown in Figure 5.6.2.

## 5.7  Rotation of Axes

Example 1.

Find the equation of the curve $xy - 4 = 0$, with respect to the new coordinate axes $X$ and $Y$ formed by a counterclockwise rotation of 30 degrees (Figure 5.7.3).

Solution：In this example,

$$\alpha = 30°, \quad \sin \alpha = \frac{1}{2}, \quad \cos \alpha = \frac{\sqrt{3}}{2}.$$

Thus $x = -\dfrac{\sqrt{3}}{2}X - \dfrac{1}{2}Y, \quad y = \dfrac{1}{2}X + \dfrac{\sqrt{3}}{2}Y.$

Substitute into the original equation and collect terms.

$$xy - 4 = 0,$$

$$(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y) \cdot (\frac{1}{2}X + \frac{\sqrt{3}}{2}Y) - 4 = 0,$$

$$\frac{\sqrt{3}}{4}X^2 + \frac{1}{2}XY - \frac{\sqrt{3}}{4}Y^2 - 4 = 0.$$

Given any second degree equation

(1)  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

and any angle of rotation $\alpha$, one can substitute the equations of rotation and collect terms to get a new second degree equation in the $X$ and $Y$ coordinates,

(2) $A_1 X^2 + B_1 XY + C_1 Y^2 + D_1 X + E_1 Y + F_1 = 0$.

It can be shown that the discriminant is unchanged by the rotation; that is,

$$B^2 - 4AC = B_1^{\ 2} - 4A_1 C_1.$$

This gives a useful check on the computations.

In Example 1 above, the original discriminant is

$$B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot 0 = 1.$$

The new equation has the same discriminant,

$$B_1^{\ 2} - 4A_1 C_1 = (\frac{1}{2})^2 - 4(\frac{\sqrt{3}}{4})(-\frac{\sqrt{3}}{4}) = \frac{1}{4} + \frac{3}{4} = 1.$$

Example 2.

Sketch the curve $xy - 4 = 0$.

Solution：

*Step 1* Apply the Discriminant Test to find the type of curve.

$$B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot 0 = 1.$$

The discriminant is positive, so the curve is a hyperbola.

*Step 2* Find an angle $\alpha$ with

$$\cot(2\alpha) = \frac{A - C}{B}.$$

$$\cot(2\alpha) = \frac{0 - 0}{1} = 0.$$

$2\alpha = 90°, \quad \alpha = 45°.$

*Step 3*    Change coordinate axes using the rotation equations.

$$\cos\alpha = \frac{\sqrt{2}}{2}, \quad \cos\alpha = \frac{\sqrt{2}}{2}.$$

$$x = X\cos\alpha - Y\sin\alpha = \frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y.$$

$$y = X\sin\alpha + Y\cos\alpha = \frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}Y.$$

Substituting, we get

$xy - 4 = 0,$

$$(\frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y)\cdot(\frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}Y) - 4 = 0,$$

$$\frac{1}{2}X^2 - \frac{1}{2}Y^2 - 4 = 0.$$

As a check, the discriminant is still $0^2 - 4\cdot(\frac{1}{2})\cdot(-\frac{1}{2}) = 1.$

*Step 4* Draw the $X$ and $Y$ axes as dotted lines and sketch the curve.

The new axes are found by rotating the old axes by $\alpha = 45°$. The curve is shown in Figure 5.7.5.

## 5.8    The ε，δ Condition for Limits

Example 1.

Consider the limit $\lim\limits_{x\to 0}(1 + \frac{10x^2}{x}) = 1.$

Solution：(%i1) limit(1+(10*x^2)/x,x,0);    極限指令：limit(方程式，極限變數，範

//方程式為 $1+\dfrac{10x^2}{x}$ ，極限變數為 x，範圍為 x 趨近於 0

(%o1)  1

When $x = 0$, the function $f(x) = 1 + 10x^2 / x$ is undefined. When $x$ is a real number close to but not equal to 0, $f(x)$ is close to 1.

Now let us be more explicit. How should we choose $x$ to get $f(x)$ strictly within $\dfrac{1}{5}$ of 1 ? To solve this problem we assume $x$ is strictly within some distance $\delta$ of 0 and get inequalities for $f(x)$.

By the lemma, we must find a $\delta > 0$ such that whenever

$-\delta < x < \delta$ and $x \neq 0$,

we have $1 - \dfrac{1}{5} < f(x) < 1 + \dfrac{1}{5}$.

Assume $-\delta < x$ and $x < \delta$.

Then $-10\delta < 10x$ and $10x < 10\delta$

$-10\delta < \dfrac{10x^2}{x}$ and $\dfrac{10x^2}{x} < 10\delta$ if $x \neq 0$

$1 - 10\delta < 1 + \dfrac{10x^2}{x}$ and $1 + \dfrac{10x^2}{x} < 1 + 10\delta$

$1 - 10\delta < f(x) < 1 + 10\delta$.

If we set $\delta = \dfrac{1}{50}$, then $1 - \dfrac{1}{5} < f(x) < 1 + \dfrac{1}{5}$.

This shows that

whenever $-\dfrac{1}{50} < x < \dfrac{1}{50}$ and $x \neq 0$, $1 - \dfrac{1}{5} < f(x) < 1 + \dfrac{1}{5}$.

In other words, when $0 < |x| < \dfrac{1}{50}$, $|f(x) - 1| < \dfrac{1}{5}$.

A similar computation shows that for each $\varepsilon > 0$ , if $0 < |x| < \varepsilon / 10$ then $|f(x) - 1| < \varepsilon$. Thus the $\varepsilon$, $\delta$ condition for $\lim_{x \to 0}(1 + 10x^2 / x) = 1$ is true, and, for a given $\varepsilon$, a corresponding $\delta$ is $\delta = \varepsilon / 10$.

Example 2.

In the limit $\lim_{x \to 2} x^2 = 4$ , find a $\delta > 0$ such that whenever

$$0 < |x - 2| < \delta, |x^2 - 4| < \frac{1}{10}.$$

Solution：(%i1) limit(x^2,x,2);  極限指令：limit(方程式，極限變數，範圍)  // 方程式為 $x^2$，極限變數為 x，範圍為 x 趨近於 2

  (%o1)  4

By the Lemma, we must find $\delta > 0$ such that whenever

$2 - \delta < x < 2 + \delta$ and $x \neq 2$,

$$4 - \frac{1}{10} < x^2 < 4 + \frac{1}{10}.$$

Assume that $2 - \delta < x$ and $x < 2 + \delta$.

As long as $2 - \delta$ and $x$ are positive we may square both sides,

$4 - 4\delta + \delta^2 < x^2$ and $x^2 < 4 + 4\delta + \delta^2$

$4 + (-4\delta + \delta^2) < x^2$ and $x^2 < 4 + (4\delta + \delta^2)$.

Now take $\delta$ small enough so that

$$-\frac{1}{10} \leq -4\delta + \delta^2 \text{ and } 4\delta + \delta^2 \leq \frac{1}{10}.$$

For example, $\delta = \frac{1}{50}$ will do. Then

$$4 - \frac{1}{10} < x^2 < 4 + \frac{1}{10}.$$

Thus whenever $0 < |x - 2| < \frac{1}{50}$, $|x^2 - 4| < \frac{1}{10}$.

Notice that any smaller value of $\delta$, such as $\delta = \frac{1}{100}$, will also work.

Example 3.

In the limit $\lim\limits_{t \to \infty} 2 + \frac{3}{t} = 2$, find a real number $B > 0$ such that whenever $t > B$, $(2 + 3/t)$ is strictly within $1/100$ of 2.

Solution：(%i1) limit(2+3/t,t,inf); 極限指令：limit(方程式，極限變數，範圍) //
方程式為 $2 + \dfrac{3}{t}$，極限變數為 t，範圍為 t 趨近於 $\infty$

  (%o1) 2

To find $B$, we assume $t > B$ and $t > 0$, and get inequalities for $2 + 3/t$.

$0 < t, \quad t > B$

$0 < \dfrac{3}{t}, \quad \dfrac{3}{t} < \dfrac{3}{B}$

$2 < 2 + \dfrac{3}{t}, \quad 2 + \dfrac{3}{t} < 2 + \dfrac{3}{B}$

Now choose $B$ so that $3?B \leq 1/100$. The number $B = 300$ will do. It follows that whenever $t > 300$,

$$2 < 2 + \frac{3}{t} < 2 + \frac{1}{100},$$

and $2 + \dfrac{3}{t}$ is strictly within $\varepsilon = \dfrac{1}{100}$ of 2.

Example 4.

In the limit $\lim_{x \to \infty}(x^2 - x) = \infty$ , find a $B > 0$ such that whenever $x > B, x^2 - x > 10000$ .

Solution：(%i1) limit(x^2-x,x,inf); 極限指令：limit(方程式，極限變數，範圍) // 方程式為 $x^2 - x$ ，極限變數為 x，範圍為 x 趨近於 $\infty$

  (%o1) $\infty$

This time we assume $x > B$ and get an inequality for $x^2 - x$. We may assume $B > 1$.

$x > B > 1$

$x - 1 > B - 1 > 0$

$x(x - 1) > B(B - 1)$

$x^2 - x > B^2 - B$ .

Now take a $B$ such that $B^2 - B > 10000$ . The number $B = 200$ will do, because $(200)^2 - 200 = 39800$. Thus whenever $x > 200, x^2 - x > 10000$.
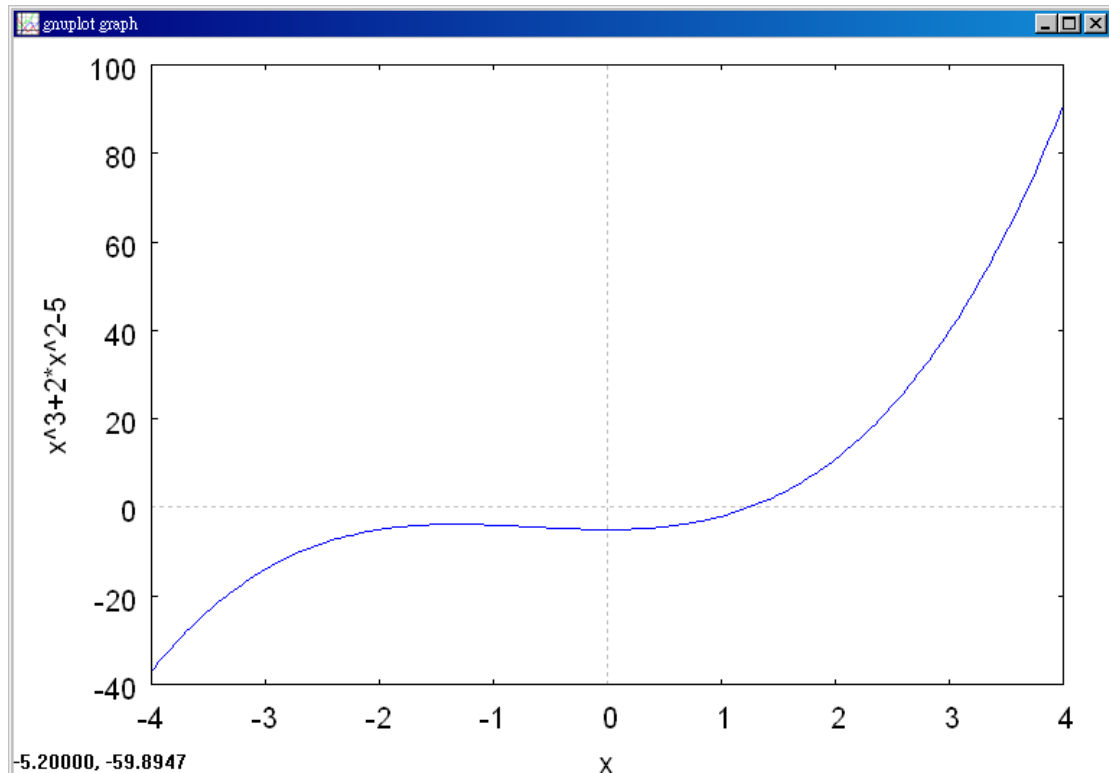
**5.9　Newton's Method**

Example 1.

Approximate a zero of $f(x) = x^3 + 2x^2 - 5$ by Newton's method.

Solution：(%i1) plot2d(x^3+2*x^2-5,[x,-4,4]); //畫出 $x^3 + 2x^2 - 5$ 的圖形，x 軸的範圍從-4~4

  (%o1)

(%i2) load(newton1);　由於 Maxima 不能直接做牛頓法，因此需要讀取 newton1 此模組，此模組提供了指令去作牛頓法　//讀取 newton1 模組

(%o2)

C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/numeric/newton1.ma

(%i3) newton(x^3+2*x^2-5,x,1,1);　牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps　//牛頓法求 $x^3 + 2x^2 - 5$ 的解，變數為 x，初始值為 x=1，函數在 x 點的估計值

(%o3)　1.285714285714286

(%i4) ev(x^2-2,x=%);　指令 ev：寫一小段獨立的程式，不受其他指令干擾　//引用前面結果算出 $f(1.285714285714286)$

(%o4)　−0.3469387755102

(%i5) newton(x^3+2*x^2-5,x,1,1/10);　牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps　//牛頓法求 $x^3 + 2x^2 - 5$ 的解，變數

為 x，初始值為 x=1，函數在 x 點的估計值

```
(%o5)   1.243001443001443
```

(%i6) ev(x^2-2,x=%);   指令 ev：寫一小段獨立的程式，不受其他指令干擾   //引用前面結果算出 $f$(1.243001443001443)

```
(%o6)  -0.45494741269633
```

(%i7) newton(x^3+2*x^2-5,x,1,1/100);   牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps   //牛頓法求 $x^3 + 2x^2 - 5$ 的解，變數為 x，初始值為 x=1，函數在 x 點的估計值

```
(%o7)   1.241897290865231
```

(%i8) ev(x^2-2,x=%);   指令 ev：寫一小段獨立的程式，不受其他指令干擾   //引用前面結果算出 $f$(1.241897290865231)

```
(%o8)  -0.4576911189416
```

*Step 1* The graph is shown in Figure 5.9.4. We choose $x_1 = 1$ as our first approximation.

*Step 2*  $f'(x) = 3x^2 + 4x$

*Step 3*  $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1 - \dfrac{(-2)}{7} = \dfrac{9}{7} \sim 1.2857$

*Step 4*  $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = x_2 - \dfrac{x_2{}^3 + 2x_2{}^2 - 5}{3x_2{}^2 + 4x_2} \sim 1.2430$

As a check we compute

$f(x_3) = x_3{}^3 + 2x_3{}^2 - 5 \sim 0.01$

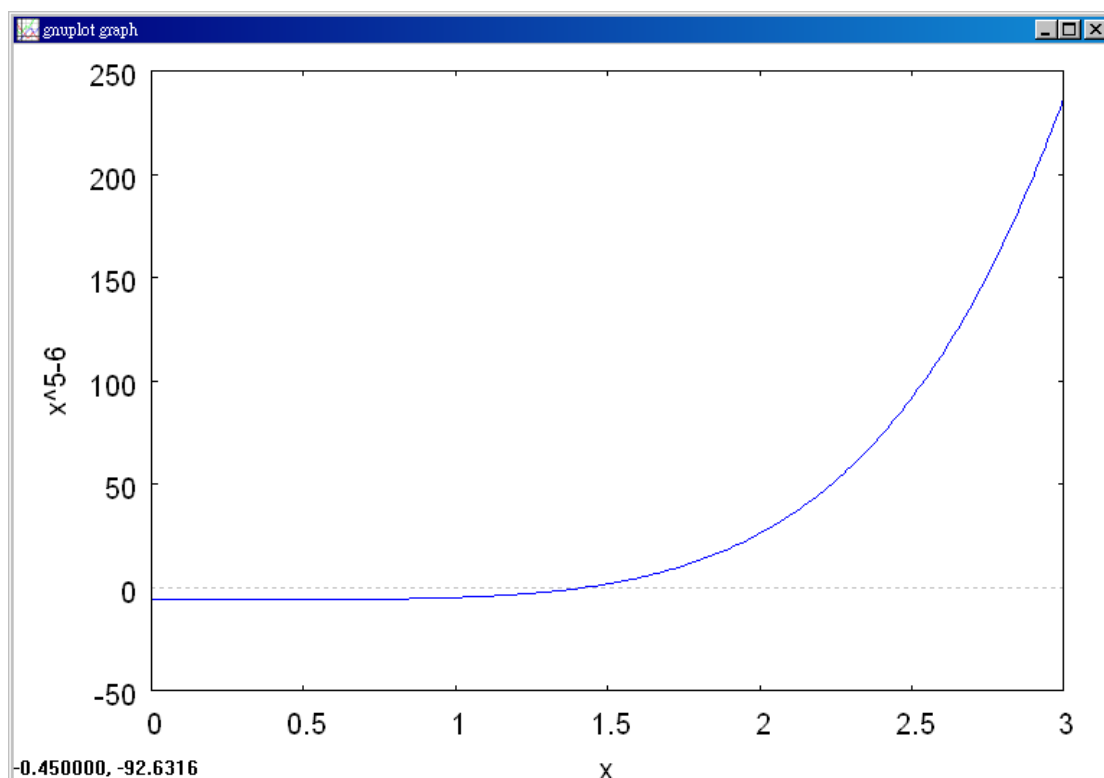One more iteration gives much more accuracy：

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_2 - \frac{x_2{}^3 + 2x_2{}^2 - 5}{3x_2{}^2 + 4x_2} \sim 1.241897$$

$$f(x_4) = x_4{}^3 + 2x_4{}^2 - 5 \sim 0.000007$$

Example 2.

Approximation the fifth root of 6 by Newton's method.

Solution：(%i1) plot2d(x^5-6,[x,0,3]);   //畫出 $x^5 - 6$ 的圖形，x 軸的範圍從 0~3

(%o1)



(%i2) load(newton1);   由於 Maxima 不能直接做牛頓法，因此需要讀取 newton1 此模組，此模組提供了指令去作牛頓法   //讀取 newton1 模組

(%o2)

C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/numeric/newton1.ma

(%i3) newton(x^5-6,x,1.5,1);    牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps    //牛頓法求 $x^5-6$ 的解，變數為 x，初始值為 x=1.5，函數在 x 點的估計值

(%o3)  1.437037037037037

(%i4) ev(x^5-6,x=%);    指令 ev：寫一小段獨立的程式，不受其他指令干擾    //引用前面結果算出 $f(1.437037037037037)$

(%o4)  0.12829703556376

(%i5) newton(x^5-6,x,1.5,1/10);    牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps    //牛頓法求 $x^5-6$ 的解，變數為 x，初始值為 x=1.5，函數在 x 點的估計值

(%o5)  1.431020109633641

(%i6) ev(x^5-6,x=%);    指令 ev：寫一小段獨立的程式，不受其他指令干擾    //引用前面結果算出 $f(1.431020109633641)$

(%o6)  0.0010698798853719

(%i7) newton(x^5-6,x,1.5,1/100);    牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps    //牛頓法求 $x^5-6$ 的解，變數為 x，初始值為 x=1.5，函數在 x 點的估計值

(%o7)  1.431020109633641

(%i8) ev(x^5-6,x=%);    指令 ev：寫一小段獨立的程式，不受其他指令干擾    //引用前面結果算出 $f(1.431020109633641)$

(%o8)  0.0010698798853719

(%i9) newton(x^5-6,x,1.5,1/1000);    牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps    //牛頓法求 $x^5-6$ 的解，變數為 x，初始值為 x=1.5，函數在 x 點的估計值

(%o9)  1.430969084744363

(%i10) ev(x^5-6,x=%);　　指令 ev：寫一小段獨立的程式，不受其他指令干擾　//
引用前面結果算出 $f$ (1.430969084744363)

(%o10)　7.6293206419109083 $10^{-8}$

(%i11) newton(x^5-6,x,1.5,1/10000);　　牛頓法指令：newton(函數，變數，變數的
初始值，eps)，函數的絕對值必須小於 eps　　//牛頓法求 $x^5 - 6$ 的解，變數為 x，
初始值為 x=1.5，函數在 x 點的估計值

(%o11)　1.430969084744363

(%i12) ev(x^5-6,x=%);　　指令 ev：寫一小段獨立的程式，不受其他指令干擾　//
引用前面結果算出 $f$ (1.4309690847744363)

(%o12)　7.6293206419109083 $10^{-8}$

*Step 1* We must find the zero of $f(x) = x^5 - 6$. The graph is shown in Figure 5.9.5.
　　　　Choose $x_1 = 1.5$.

*Step 2* $f'(x) = 5x^4$

*Step 3* $x_2 = x_1 - \dfrac{x_1^5 - 6}{5x_1^4} \sim 1.437$

*Step 4* $x_3 = x_2 - \dfrac{x_2^5 - 6}{5x_2^4} \sim 1.43102$

As a check we compute $(x_3)^5 \sim 6.001$

In this example more iterations would be necessary if our first approximation had not
been chosen as well. For instance, starting with $x_1 = 1$ we would not reach the
approximation 1.431 until $x_6$, obtaining the successive approximations

$x_1 = 1$, $x_2 = 2$, $x_3 = 1.675$, $x_4 = 1.49245$, $x_5 = 1.43583$, $x_6 = 1.43100$.
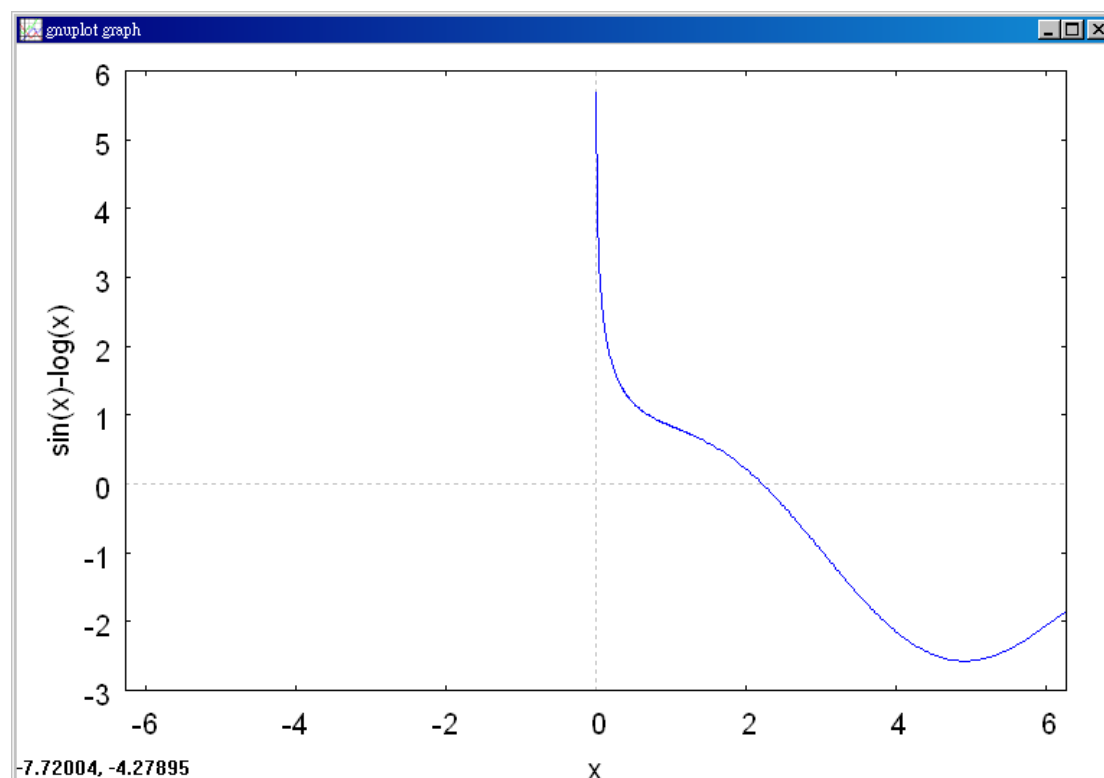
Example 3.

Approximate the point  $x$  where  $\sin x = \ln x$.

Solution：(%i1) plot2d(sin(x)-log(x),[x,-2*%pi,2*%pi]);   //畫出 $\sin x = \ln x$ 的圖形，x 軸的範圍從 $-2\pi \sim 2\pi$

```
plot2d: expression evaluates to non-numeric value somewhere in plott
  (%o1)
```



(%i2) load(newton1);   由於 Maxima 不能直接做牛頓法，因此需要讀取 newton1 此模組，此模組提供了指令去作牛頓法   //讀取 newton1 模組

```
  (%o2)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/numeric/newton1.ma
```

(%i3) newton(sin(x)-log(x),x,2,1);   牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps   //牛頓法求 $\sin(x) = \ln x$ 的解，變數為 x，初始值為 x=2，函數在 x 點的估計值

```
  (%o3)  2
```

(%i4) float(ev(sin(x)-log(x),x=%));   指令 ev：寫一小段獨立的程式，不受其他指

```
(%o4)  0.21615024626574
```

(%i5) newton(sin(x)-log(x),x,2,1/10);　牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps　　//牛頓法求 $\sin(x) = \ln x$ 的解，變數為 x，初始值為 x=2，函數在 x 點的估計值

```
(%o5)  2.235934063889129
```

(%i6) float(ev(sin(x)-log(x),x=%));　指令 ev：寫一小段獨立的程式，不受其他指令干擾　//引用前面結果算出 $f(20235934063889129)$

```
(%o6)  -0.017827280704095
```

(%i7) newton(sin(x)-log(x),x,2,1/100);　牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps　　//牛頓法求 $\sin(x) = \ln x$ 的解，變數為 x，初始值為 x=2，函數在 x 點的估計值

```
(%o7)  2.219185521531423
```

(%i8) float(ev(sin(x)-log(x),x=%));　指令 ev：寫一小段獨立的程式，不受其他指令干擾　//引用前面結果算出 $f(2.219185521531423)$

```
(%o8)  -8.2643594711551138 10⁻⁵
```
$$(\%o8)\ -8.2643594711551138 \times 10^{-5}$$

(%i9) newton(sin(x)-log(x),x,2,1/1000);　牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps　　//牛頓法求 $\sin(x) = \ln x$ 的解，變數為 x，初始值為 x=2，函數在 x 點的估計值

```
(%o9)  2.219185521531423
```

(%i10) float(ev(sin(x)-log(x),x=%));　指令 ev：寫一小段獨立的程式，不受其他指令干擾　//引用前面結果算出 $f(2.219185521531423)$

$$(\%o10)\ -8.2643594711551138 \times 10^{-5}$$

As one can see from the graphs of $\sin x$ and $\ln x$ in figure 5.9.6, $\sin x$ and $\ln x$ cross at one point $x$, which is somewhere between $x = 1$ (where $\ln x$ crosses the

$x$-axis going up) and $x = \pi$ (where $\sin x$ crosses the $x$-axis going down). To apply Newton's method, we let $f(x)$ be the function

$$f(x) = \sin x - \ln x$$

shown in Figure 5.9.7. We wish to approximate the zero of $f(x)$.

*Step 1* Choose $x_1 = 2$ (since the zero of $f(x)$ is between 1 and $\pi$).

*Step 2*  $f'(x) = \cos x - 1/x$

*Step 3*  $x_2 = x_1 - \dfrac{\sin x_1 - \ln x_1}{\cos x_1 - 1/x_1} = 2 - \dfrac{\sin 2 - \ln 2}{\cos 2 - 1/2} \sim 2.23593$

*Step 4* Repeat Step 3. The values of $x_n$, $f(x_n)$, and $f'(x_n)$ are shown in the table.

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ |
|---|---|---|---|
| 1 | 2.000000000 | 0.216150246 | -0.916146836 |
| 2 | 2.235934064 | -0.017827280 | -1.064407894 |
| 3 | 2.219185522 | -0.000082645 | -1.054519059 |
| 4 | 2.219107150 | -0.000000001 | -1.054472505 |

The answer is  $x \sim 2.219107150$.

On a calculator we find that

$$\sin(2.219107150) = 0.797104929$$

$$\ln(2.219107150) = 0.797104930.$$

## 5.10  Derivatives and Increments

Example 1.

Consider the curve  $f(x) = x^{1/3}$.

Then  $f'(x) = \dfrac{1}{3} x^{-2/3}$.

At the point $x = 8$, we have

$$x = 8, \quad f(x) = 2, \quad f'(x) = \frac{1}{12} = 0.0833....$$

Thus $\lim\limits_{\Delta x \to 0} \dfrac{(8 + \Delta x)^{1/3} - 2}{\Delta x} = \dfrac{1}{12}.$

This is the slope of the line tangent to the curve at the point (8, 2). As $\Delta x$ approaches zero, the slope of the secant line through the two points (8, 2) and $(8 + \Delta x, (8 + \Delta x)^{1/3})$ will approach $\dfrac{1}{12}$. We make a table showing the slope of the secant line for various values of $\Delta x$.

| $\Delta x$ | $\Delta y = (8 + \Delta x)^{1/3} - 2$ | $\dfrac{\Delta y}{\Delta x} = $ slope of secant line | $\left\| \dfrac{\Delta y}{\Delta x} - \dfrac{1}{12} \right\|$ |
|---|---|---|---|
| 10 | 0.6207 | 0.0621 | 0.0212 |
| 1 | 0.0801 | 0.0801 | 0.0032 |
| $\dfrac{1}{10}$ | 0.00829 | 0.0830 | 0.0003 |
| -10 | -3.2599 | 0.3260 | 0.2427 |
| -1 | -0.0871 | 0.0871 | 0.0038 |
| $-\dfrac{1}{10}$ | -0.00837 | 0.0837 | 0.0004 |

Example 1. (Concluded)

We consider once more the curve $f(x) = x^{1/3}$ at the point $x = 8$. The second derivative is

$$f''(x) = -\frac{2}{9} x^{-5/3}.$$

First consider the interval [8, 9]. In this interval $f''(x)$ has the maximum value

$$\left| f''(8) \right| = \frac{2}{9}(8)^{-5/3} = \frac{2}{9} 2^{-5} = \frac{1}{144}.$$

Thus we may take $M = \dfrac{1}{144}$, and

$\dfrac{1}{2} M \Delta x = \dfrac{1}{288} \Delta x$ is an error estimate for $\lim\limits_{\Delta x \to 0^{+}} \dfrac{\Delta y}{\Delta x} = \dfrac{1}{12}$.

Thus when $\Delta x = 1$, $\left| \dfrac{\Delta y}{\Delta x} - \dfrac{1}{12} \right| \leq \dfrac{1}{288} = 0.0035$,

when $\Delta x = \dfrac{1}{10}$, $\left| \dfrac{\Delta y}{\Delta x} - \dfrac{1}{12} \right| \leq \dfrac{1}{2880} = 0.00035$.

Next consider the interval [7, 8]. This time we take

$M = \left| f''(7) \right| = \dfrac{2}{9} (7)^{-5/3} = 0.0087$.

Then $\dfrac{1}{2} M |\Delta x| = 0.0044 |\Delta x|$

is an error estimate for the limit

$\lim\limits_{\Delta x \to 0^{-}} \dfrac{\Delta y}{\Delta x} = \dfrac{1}{12}$.

when $\Delta x = -1$, $\left| \dfrac{\Delta y}{\Delta x} - \dfrac{1}{12} \right| \leq 0.0044$,

when $\Delta x = -\dfrac{1}{10}$, $\left| \dfrac{\Delta y}{\Delta x} - \dfrac{1}{12} \right| \leq 0.00044$.

From the table in Example 1 we see that the error estimates are slightly greater than the actual values of $\left| \dfrac{\Delta y}{\Delta x} - \dfrac{1}{12} \right|$.

Example 2.

Find approximate values for $\sqrt[3]{9}$ and $\sqrt[3]{7.9}$. Both these numbers are close to 8, whose cube root 2 comes out even. Taking $f(x) = \sqrt[3]{x}$ and $c = 8$, we have

$f(c) = 2, \quad f'(c) = \dfrac{1}{12} = 0.0833....$

From Theorem 1 the approximate values are

$f(c + \Delta x) \sim f(c) + f'(c)\Delta x$.

Thus

$\sqrt[3]{9} \sim 2 + \dfrac{1}{12} = 2.0833$,

$\sqrt[3]{7.9} \sim 2 + \dfrac{1}{12}(-0.1) = 1.99167$.

To get an error estimate for $\sqrt[3]{9}$, take the interval [8, 9]. From Example 1 we may take $M = \dfrac{1}{144}$. Therefore by Theorem 1,

$\sqrt[3]{9} \sim 2.0833, \quad \text{error} \leq \dfrac{1}{2} \cdot \dfrac{1}{144} \cdot 1^2 = 0.0035$.

Thus $2.0798 \leq \sqrt[3]{9} \leq 2.0868$.

To get an error estimate for $\sqrt[3]{7.9}$ take the interval [7, 8] and $M = 0.0087$.

By Theorem 1,

$\sqrt[3]{7.9} \sim 1.991667, \text{error} \leq \dfrac{1}{2}(0.0087)(0.1)^2 = 0.000044$.

Thus $1.991623 \leq \sqrt[3]{7.9} \leq 1.991711$.

Example 3.

Find an approximate value for $(0.99)^5$.

Let $f(x) = x^5, \quad c = 1$.

Then $f(c) = 1^5 = 1, \quad f'(c) = 5c^4 = 5$.

We put $0.99 = c + \Delta x$, $\Delta x = -0.01$.

Then the approximate value is

$f(c + \Delta x) \sim f(c) + f'(c)\Delta x$,

$(0.99)^5 \sim 1 + 5(-0.01) = 0.95$.

To get an error estimate we see that $f''(u) = 20u^3$, so $|f''(u)| \leq 20$ for $u$ between 0.99 and 1. Then $M = 20$, and

$(0.99)^5 \sim 0.95$, error $\leq \dfrac{(0.01)^2}{2}(20) = 0.001$,

or $0.949 \leq (0.99)^5 \leq 0.951$.