4.1 THE DEFINITE INTEGRAL

Example 1

Let $f(x)=\frac{1}{2}x$. In Figure4.1.6, the region under the curve from $x=0$ to $x=2$ is a triangle with base 2 and height 1, so its area should be

$$A=\frac{1}{2}bh=1$$

(%i5) integrate(1/2*((x)), x, 0, 2);

(%o5) 1

函數爲 $f(x)=\frac{1}{2}x$,區間範圍從 0 到 2,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Let us compare this value for the area with some Riemann sums. In Figure 4.1.7, we take $\Delta x=\frac{1}{2}$. The interval [0, 2] divides into four subintervals $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$, $[1,\frac{3}{2}]$, and $[\frac{3}{2},2]$. We make a table of values of $f(x)$ at the lower endpoints.

x_k	0	$\frac{1}{2}$	1	$\frac{3}{2}$
$f(x_k)$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$

The Riemann sum is then

$$\sum_{0}^{2} f(x)\Delta x = 0\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{3}{4}\cdot\frac{1}{2}=\frac{6}{8}.$$

(%i5) lsum (f(x)/2, x, [0, 1/2, 1,3/2]);

(%o5) $\frac{3}{4}$

將0到2進行 $\frac{1}{2}$ 的切割,並代入 $f(x)=\frac{1}{2}x$,再連加求總和

可用指令lsum(函數,變數,欲代入的值);

In Figure 4.1.8, we take $\Delta x = \frac{1}{4}$. The table of values is as follows.

x_k	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$
$f(x_k)$	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$

The Riemann sum is

$$\sum_{0}^{2} f(x)\Delta x = 0\cdot\frac{1}{4}+\frac{1}{8}\cdot\frac{1}{4}+\frac{2}{8}\cdot\frac{1}{4}+\frac{3}{8}\cdot\frac{1}{4}+\frac{4}{8}\cdot\frac{1}{4}+\frac{5}{8}\cdot\frac{1}{4}+\frac{6}{8}\cdot\frac{1}{4}+\frac{7}{8}\cdot\frac{1}{4}=\frac{7}{8}$$

(%i6) lsum (f(x)/4, x, [0, 1/4, 2/4,3/4,4/4,5/4,6/4,7/4]);

(%o6) $\frac{7}{8}$

將0到2進行$\frac{1}{4}$的切割,並代入 $f(x)=\frac{1}{2}x$,再連加求總和

可用指令lsum(函數,變數,欲代入的值);

We see that the value is getting closer to one.

Finally, let us take a value of Δx that does not divide evenly into the interval length 2. Let $\Delta x = 0.6$. We see in Figure 4.1.9 that the interval then divides into three subintervals of length 0.6 and one of length 0.2, namely [0, 0.6], [0.6, 1.2], [1.2, 1.8], [1.8, 2.0].

x_k	0	0.6	1.2	1.8
$f(x_k)$	0	0.3	0.6	0.9

The Riemann sum is

$$\sum_{0}^{2} f(x)\Delta x = 0(.6)+(.3)(.6)+(.6)(.6)+(.9)(.2)=.72.$$

(%i8) lsum (f(x)*0.6, x, [0, 0.6, 1.2]);

(%o8) 0.54

http://www.npue.edu.tw/academic/math/index.htm

可用指令lsum(函數,變數,欲代入的值);

(%i10) solve[%o8+0.9*0.6/3];

(%o10) $solve_{0.72}$

將0到2進行0.6的切割,並代入 $f(x)=\frac{1}{2}x$,再連加求總和

最後一段切割部分,因為1.8到2只佔了0.6的1/3,所以面積要除以3

Example 2 Let $f(x)=\sqrt{1-x^2}$, defined on the closed interval I =[-1, 1]. The region under the curve is a semicircle of radius 1. We know from plane geometry that the area is $\pi/2$, or approximately 3.14/2=1.57. Let us compute the values of some Riemann sums for this function to see how close they are to 1.57. First take $\Delta x=\frac{1}{2}$ as in Figure 4.1.10(a). We make a table of values.

x_k	-1	-1/2	0	1/2
$f(x_k)$	0	$\sqrt{3/4}$	1	$\sqrt{3/4}$

The Riemann sum is then.

$$\sum_{-1}^{1} f(x)\Delta x = 0\cdot 1/2 +.\sqrt{3/4}\cdot 1/2 + 1\cdot 1/2 + \sqrt{3/4}\cdot 1/2$$

$$=\frac{1+\sqrt{3}}{2}\sim 1.37$$

將-1到1進行$\frac{1}{2}$的切割,並代入 $f(x)=\sqrt{1-x^2}$,再連加求總和

Next we take $\Delta x=\frac{1}{5}$ .Then the interval [-1, 1] is divided into ten subintervals as in Figure 4.1.10(b). Our table of values is as follows.

x_k	-1	$-\frac{4}{5}$	$-\frac{3}{5}$	$-\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
$f(x_k)$	0	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{\sqrt{21}}{5}$	$\frac{\sqrt{24}}{5}$	1	$\frac{\sqrt{24}}{5}$	$\frac{\sqrt{21}}{5}$	$\frac{4}{5}$	$\frac{3}{5}$

The Riemann sum is

$$\sum_{-1}^{1} f(x)\Delta x = \frac{1}{5}\left[0+\frac{3}{5}+\frac{4}{5}+\frac{\sqrt{21}}{5}+\frac{\sqrt{24}}{5}+1+\frac{\sqrt{24}}{5}+\frac{\sqrt{21}}{5}+\frac{4}{5}+\frac{3}{5}\right].$$

$$= \frac{19+2\sqrt{21}+2\sqrt{24}}{25} \sim 1.52$$

Thus we are getting closer to the actual area

將-1到1進行$\frac{1}{5}$的切割,並代入 $f(x)=\sqrt{1-x^2}$,再連加求總和

(%i3) integrate(sqrt((1-x^2)), x, -1, 1);

(%o3) $\frac{\pi}{2}$

函數爲 $f(x)=\sqrt{1-x^2}$,區間範圍從-1 到 1,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 3   Given a constant $c>0$, evaluate the integral $\int_a^b c\ dx$.

Figure 4.1.13 shows that for every positive real number Δx ,the finite Riemann sum is

$$\sum_a^b c\,\Delta x = c(b-a)$$

By the Transfer Principle, the infinite Riemann sum in Figure 4.1.14 has the same value, $\sum_a^b c\,dx = c(b-a)$.

Taking standard parts,

$$\int_a^b c\,dx = c(b-a).$$

This is the familiar formula for the area of a rectangle.

(%i2) integrate(c, x, a, b);

(%o2) $(b-a)\,c$

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函數爲 c ,區間範圍從 a 到 b,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 4 Given $b > 0$, evaluate the integral $\int_0^b x\,dx$

The area under the line $y = x$ is divided into vertical strips of width dx.
Study Figure 4.1.15. The area of the lower region A is the infinite Riemann sum

(1) area of $A = \sum_0^b x\,dx$

By symmetry, the upper region B has the same area as A;

(2) area of $A =$ area of B

Call the remaining region C, formed by the infinitesimal squares along the diagonal. Thus

(3) area of $A +$ area of $B +$ area of $C = b^2$

Each square in C has height dx except the last one, which may be smaller, and the widths add up to b, so

(4) $0 \le$ area of C $\le$ b dx

Putting (1)-(4) together,

$$2\sum_0^b x\ dx \le b^2 \le \left(2\sum_0^b x\ dx\right) + bx$$

Sine $b\ dx$ is infinitesimal,

$$2\sum_0^b x\ dx \approx b^2$$

$$\sum_0^b x\ dx \approx \frac{b^2}{2}$$

Taking standard parts, we have

$$\int_0^b x\ dx = \frac{b^2}{2}$$

(%i1) integrate(x, x, 0, b);

(%o1) $\frac{b^2}{2}$

函數爲 $f(x) = \sqrt{1 - x^2}$,區間範圍從-1到1,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

4.2 FUNDAMENTAL THEOREM OF CALCULUS

Example 1

(a) Find $\int_a^b c\ dx$. Since cx is an antiderivative of c

$$\int_a^b c\ dx = cb - ca = c(b-a).$$

(b) Find $\int_a^b x\ dx$. $\frac{1}{2}x^2$ is an antiderivative of x .Thus

$$\int_a^b x\ dx = \frac{1}{2}b^2 - \frac{1}{2}a^2.$$

The above example gives the same result that we got before but is much simpler. We can easily go further.

(%i6) integrate(c, x, a, b);

(%o6) $(b-a)c$

函數爲 c ,區間範圍從 a 到 b ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

(%i7) integrate(x, x, a, b);

(%o7) $\frac{b^2}{2} - \frac{a^2}{2}$

函數爲 x ,區間範圍從 a 到 b ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 2 Find $\int_a^b x^2\,dx$. $x^3/3$ is an antiderivative of x^2 because

$$\frac{d(x^3/3)}{dx}=\frac{3x^2}{3}=x^2$$

Therefore $$\int_a^b x^2\,dx=\frac{b^3}{3}-\frac{a^3}{3}$$

This gives the area of the region under the curve $y=x^2$ between a and b (Figure 4.2.9).

(%i8) integrate(x^2, x, a, b);

(%o8) $\frac{b^3}{3}-\frac{a^3}{3}$

函數為 x^2 ,區間範圍從 a 到 b ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 3 A particle moves along the y-axis with velocity $v=8t^3$ cm/sec. How far does it move between times $t=-1$ and $t=2$ sec? The function $G(t)=2t^4$ is an antiderivative of the velocity $v=8t^3$. Thus the definite integral is

$$\text{distance moved}=\int_{-1}^{2}8t^3\,dt=2\cdot 2^4-2\cdot(-1)^4=30\ cm$$

(%i9) integrate(8*t^3, t, -1, 2);

(%o9) 30

函數為 $v=8t^3$,區間範圍從-1 到 2 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 4 Find $\int_0^4 \sqrt{t}\,dt$ .The function $\sqrt{t}$ is defined and continuous on the half-open interval $[0,\infty)$.But to apply the Fundamental. Theorem we need a function continuous on an open interval that contains the limit points 0 and 4. We therefore define

$$f(t)=\begin{cases}0 & for\ t<0\\ \sqrt{t} & for\ t\geq 0\end{cases}$$

This function is continuous on the whole real line. In particular it is continuous at 0 because if $t\approx 0$ then $f(t)\approx 0$ .The function

$$F(t)=\begin{cases}0 & for\ t<0\\ \frac{2}{3}t^{3/2} & for\ t\geq 0\end{cases}$$

is an antiderivative of .Then

$$\int_0^4 \sqrt{t}\ dt = F(4)-F(0)=(\frac{2}{3}\cdot 4^{3/2}-\frac{2}{3}\cdot 0^{3/2})=\frac{16}{3}$$

(%i1) integrate(sqrt(t), t, 0, 4);

(%o1) $\frac{16}{3}$

函數爲 $\sqrt{t}$,區間範圍從 0 到 4 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 5 The only way we can show that the function $f(x)=\sqrt{1+x^4}$ has an antiderivative is to take a definite integral

$$\int_0^x \sqrt{1+t^4}\ dt$$

This is a “new” function that cannot be expressed in terms of algebraic, trigonometric, and exponential functions without calculus.

Example 6 Let $y = \int_x^2 \sqrt{1+t^2}\, dt$. Then $y = -\int_2^x \sqrt{1+t^2}\, dt$

and $$dy = -d\left(\int_2^x \sqrt{1+t^2}\, dt\right) = -\sqrt{1+x^2}\, dx$$

Example 7 Let $y = \int_3^{x^2+x} \frac{1}{t^3+1}\, dt$

Let $u = x^2 + x$ .Then

$$\frac{du}{dx} = (2x+1), \qquad y = \int_3^u \frac{1}{t^3+1} dt, \qquad \frac{dy}{du} = \frac{1}{u^3+1}$$

By the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u^3+1}(2x+1) = \frac{2x+1}{(x^2+x)^3+1}$$

4.3 INDEFINITE INTEGRALS

Example 1 $\int (2x^{-1} + 3\sin x)\,dx = 2\ln|x| - 3\cos x + C$

We can use the rules to write down at once the indefinite integral of any polynomial.

(%i3) integrate(2*x^(-1)+3*sin(x),x);

(%o3) 2 log(x)-3 cos(x)

函數爲 $2x^{-1} + 3\sin x$,非定積分故無區間範圍,可使用指令

integrate(函數, 變數);

Example 2 $\int (4x^3 - 6x^2 + 2x + 1)\,dx = x^4 - 2x^3 + x^2 + x + C$

(%i2) integrate(4*x^3-6*x^2+2*x+1, x);

(%o2) $x^4 - 2\,x^3 + x^2 + x$

函數爲 $4x^3 - 6x^2 + 2x + 1$,非定積分故無區間範圍,可使用指令

integrate(函數, 變數);

Example 3 $\int \left(\frac{3}{x^2} + \sqrt{x} \right) dx = -\frac{3}{x} + \frac{2}{3} x^{3/2} + C$

(%i1) integrate((3/(x^2))+sqrt(x), x);

(%o1) $\frac{2\,x^{3/2}}{3} - \frac{3}{x}$

函數爲 $\frac{3}{x^2} + \sqrt{x}$,非定積分故無區間範圍,可使用指令

integrate(函數, 變數);

Example 4 Show that $\int \frac{dx}{(1+x)^{1/2}(1-x)^{3/2}} = \sqrt{\frac{1+x}{1-x}} + C$

Our rules give no hint on finding this integral. However, once the answer is given to us we can easily prove that it is correct by differentiating,

$$\frac{d\sqrt{\frac{1+x}{1-x}}}{dx} = \frac{d\left((1+x)^{1/2}(1-x)^{-1/2}\right)}{dx}$$

$$= (1+x)^{1/2}(-1)(-\frac{1}{2})(1-x)^{-3/2} + (1-x)^{-1/2}(\frac{1}{2})(1+x)^{-1/2}$$

$$= (1+x)^{-1/2}(1-x)^{-3/2}[\frac{1}{2}(1+x) + \frac{1}{2}(1-x)]$$

$$= \frac{1}{(1+x)^{1/2}(1-x)^{3/2}}$$

Example 5 A particle moves with velocity $v = 1/t^2$, $t > 0$. At time $t = 2$ it is at position $y = 1$. Find the position y as a function of t. We compute

$$\int v\, dt = \int \frac{1}{t^2} dt = -\frac{1}{t} + C$$

Since $dy/dt = v$, y is one of the functions in the family $-1/t + C$. We can find the constant C by setting $t = 2$ and $y = 1$,

$$y = -\frac{1}{t} + C, \qquad 1 = -\frac{1}{2} + C, \qquad C = 1\frac{1}{2}.$$

Then the answer is

$$y = -\frac{1}{t} + 1\frac{1}{2},$$

Example 6 Evaluate the definite integral of $y = (1+t)/t^3$ from $t = 1$ to $t = 2$ (see Figure 4.3.3)

$$\int_1^2 \frac{1+t}{t^3} dt = \int_1^2 (t^{-3} + t^{-2})\, dt$$

$$= \int_1^2 t^{-3}\, dt + \int_1^2 t^{-2} dt = \frac{t^{-2}}{-2}\Bigg]_1^2 + \frac{t^{-1}}{-1}\Bigg]_1^2$$

$$=\left(\frac{1}{(-2)\cdot 4}-\frac{1}{(-2)\cdot 1}\right)+\left(\frac{1}{-2}-\frac{1}{-1}\right)=\frac{3}{8}+\frac{1}{2}=\frac{7}{8}.$$

Thus the area under the curve $y=(1+t)/t^3$ from $t=1$ to $t=2$ is $\frac{7}{8}$.

(%i4) integrate((1+t)/t^3,t,1,2);

(%o4) $\frac{7}{8}$

函數爲 $\frac{1+t}{t^3}$,區間範圍從 1 到 2 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 7 Find the area of the region under one arch of the curve $y=\sin x$ (see Figure 4.3.4)

One arch of the sine curve is between $x=0$ and $x=\pi$ .The area is the definite integral

$$\int_0^{\pi}\sin x\,dx=-\cos x\Big]_0^{\pi}$$

$$=-\cos\pi-(-\cos 0)=-(-1)-(-1)=2.$$

The area is exactly 2.

(%i1) integrate(sin(x),x,0,%pi);

(%o1) 2

函數爲 $\sin x$,區間範圍從 0 到 π ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 8 Find the area under the curve $y=-2x^{-1}$ from $x=-5$ to $x=-1$. (see Figure 4.3.5.)

The area is given by the definite integral

$$\int_{-5}^{-1}-2x^{-1}dx$$

First compute the indefinite integral

$$\int -2x^{-1}dx = -2\int x^{-1}dx = -2\ln|x| + C.$$

Now compute the definite integral

$$\int_{-5}^{-1} -2x^{-1}dx = -2\ln|x|\Big]_{-5}^{-1}$$

$$= -2(\ln|-1| - \ln|-5|) = -2(\ln 1 - \ln 5)$$

$$= 2\ln 5 \sim 3.219.$$

(%i2) integrate(-2*x^(-1),x,-5,-1);

```
(%o2) 2 log(5)
```

函數爲 $-2x^{-1}$,區間範圍從-5 到-1 ,可使用指令

integrate(函數，變數，範圍最小值，範圍最大值);

Example 9 In computing definite integrals one must first make sure that the function to be integrated is continuous on the interval. For instance,

$$\int_{-1}^{1} \frac{1}{x^2}dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^{1} = -1-(-(-1)) = -2$$

This is clearly wrong because $1/x^2 > 0$ so the area under the curve cannot be negative. The mistake is that $1/x^2$ is undefined at $x = 0$ and hence the function is discontinuous at $x = 0$ .Therefore the area under the curve and the definite integral

$$\int_{-1}^{1} \frac{1}{x^2}dx$$

are undefined (Figure 4.3.6).

(%i5) integrate((1/x^2),x,-1,1);

```
defint: integral is divergent.
 -- an error.  To debug this try debugmode(true);
```

4.4 INTEGRATION BY CHANGE OF VARIABLES

Example1 Find $\int (4x+1)^3+(4x+1)^2+(4x+1)\,dx$. Let $u=4x+1$.

Then $du=4dx,\ dx=\frac{1}{4}du.$ Hence

$$\int (4x+1)^3+(4x+1)^2+(4x+1)\,dx$$

$$=\int (u^3+u^2+u)\cdot\frac{1}{4}du=\frac{1}{4}\left(\frac{u^4}{4}+\frac{u^3}{3}+\frac{u^2}{2}\right)+C$$

$$=\frac{1}{4}\left[\frac{(4x+1)^4}{4}+\frac{(4x+1)^3}{3}+\frac{(4x+1)^2}{2}\right]+C$$

```
(%i6) integrate((4*x+1)^3+(4*x+1)^2+(4*x+1), x);
```

(%o6) $16x^4+\frac{64x^3}{3}+12x^2+3x$

```
(%i7) integrate(u^3+u^2+u, u);
```

(%o7) $\frac{u^4}{4}+\frac{u^3}{3}+\frac{u^2}{2}$

函數爲 $(4x+1)^3+(4x+1)^2+(4x+1)$,非定積分故無區間範圍,可使用指令

integrate(函數, 變數);

Example2 Find $\int \frac{-1}{x^2(1+1/x)^2}dx.$

Let $u=1+1/x$. Then $du=-1/x^2dx$

and thus $\frac{-1}{x^2(1+1/x)^2}dx=\frac{1}{u^2}du.$

So $\int \frac{-1}{x^2(1+1/x)^2}dx=\int \frac{1}{u^2}du=\frac{u^{-1}}{-1}+C.=-\frac{1}{1+1/x}+C$

(%i8) integrate(-1/((x^2)*(1+1/x)^2), x);

(%o8) $-\frac{1}{\frac{1}{x}+1}$

函數為 $\frac{-1}{x^2(1+1/x)^2}$,非定績分故無區間範圍,可使用指令

integrate(函數, 變數);

Example3 Find $\int(1+5x)^2\,dx$. Let $u=1+5x$. For emphasis we shall do it correctly and incorrectly.

Correct: $du=5dx$, $dx=\frac{1}{5}du$,

$$\int(1+5x)^2\,dx=\int u^2\cdot\frac{1}{5}du=\frac{u^3}{15}+C=\frac{(1+5x)^3}{15}+C$$

Incorrect: $\int(1+5x)^2\,dx=\int u^2du=\frac{u^3}{3}+C=\frac{(1+5x)^3}{3}+C$

(%i9) integrate((1+5*x)^2, x);

(%o9) $\frac{25\,x^3}{3}+5\,x^2+x$

函數為 $(1+5x)^2$,非定績分故無區間範圍,可使用指令

integrate(函數, 變數);

Example4 Find $\int x^3\sqrt{2-x^2}\,dx$. Let $u=2-x^2$, $du=-2xdx$, $dx=du/(-2x)$.

We try to express the integral in terms of u.

$$\int x^3\sqrt{2-x^2}\,dx.=\int x^3\sqrt{u}\,\frac{du}{-2x}=\int-\frac{1}{2}x^2\sqrt{u}\,du.$$

Since $u=2-x^2, x^2=2-u$. Therefore

$$\int-\frac{1}{2}x^2\sqrt{u}\,du=\int-\frac{1}{2}(2-u)\sqrt{u}\,du=\int-\sqrt{u}+\frac{1}{2}u^{3/2}du$$

$$= -\frac{2}{3}u^{3/2} + \frac{1}{2}\cdot\frac{2}{5}u^{5/2} + C$$

$$= -\frac{2}{3}(2-x^2)^{3/2} + \frac{1}{5}(2-x^2)^{5/2} + C.$$

(%i10) integrate(x^3*sqrt(2-x^2), x);

(%o10) $\frac{x^2(2-x^2)^{3/2}}{5} - \frac{4(2-x^2)^{3/2}}{15}$

函數爲 $x^3\sqrt{2-x^2}$,非定績分故無區間範圍,可使用指令

integrate(函數, 變數);

Example5 Find the area under the line $y = 1+3x$ from $x = 0$ to $x = 1$. This can be done either with or without a change of variables.

Without change of variable: $\int(1+3x)dx = x + 3x^2/2 + C$, so

$$\int_0^1(1+3x)dx = x + \frac{3x^2}{2}\Big]_0^1 = \left(1+\frac{3\cdot 1^2}{2}\right) - \left(0+\frac{3\cdot 0^2}{2}\right) = \frac{5}{2}.$$

With change of variable: Let $u = 1+3x$. Then $du = 3dx$, $dx = \frac{1}{3}du$.

When $x = 0$, $u = 1+3\cdot 0 = 1$. When $x = 1$, $u = 1+3\cdot 1 = 4$.

$$\int_0^1(1+3x)dx = \int_1^4 u\cdot\frac{1}{3}du = \frac{u^2}{6}\Big]_1^4 = \frac{16}{6} - \frac{1}{6} = \frac{15}{6} = \frac{5}{2}.$$

(%i11) integrate(1+3*x, x,0,1);

(%o11) $\frac{5}{2}$

函數爲 $1+3x$,區間範圍從 0 到 1 ,可使用指令

integrate(函數, 變數, 範圍最小値, 範圍最大値);

Example6 Find the area under the curve $y = 2x/(x^2-3)^2$ from $x = 2$ to $x = 3$

Let $u = x^2 - 3$. Then $du = 2x\,dx$. At $x = 2$, $u = 2^2 - 3 = 1$. At $x = 3$, $u = 3^2 - 3 = 6$. Then

$$\int_2^3 \frac{2x}{(x^2-3)^2}dx = \int_1^6 \frac{1}{u^2}du = -\frac{1}{u}\Big]_1^6 = 1 - \frac{1}{6} = \frac{5}{6}.$$

(%i12) integrate(2*x/(x^2-3)^2, x,2,3);

(%o12) $\frac{5}{6}$

函數爲 $\frac{2x}{(x^2-3)^2}$,區間範圍從 2 到 3 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example7 Find $\int_0^1 \sqrt{1-x^2}\,x\,dx$ .The function $\sqrt{1-x^2}\,x$ as given is only defined on the closed interval [-1, 1] .In order to use Theorem2, we extend it to the open interval $J = (-\infty, \infty)$ by

$$h(x) = \begin{cases} 0 & \text{if } x < 1 \text{ or } x > 1 \\ \sqrt{1-x^2}\,x & \text{if } -1 \le x \le 1 \end{cases}$$

Let $u = 1 - x^2$. Then $du = -2x\,dx$, $dx = -du/2x$. At $x = 0, u = 1$. At $x = 1, u = 0$. Therefore

$$\int_0^1 \sqrt{1-x^2}\,x\,dx = \int_1^0 \sqrt{u}\cdot(-\frac{1}{2}du) = \int_1^0 -\frac{1}{2}\sqrt{u}\,du$$

$$= \frac{1}{2}\int_0^1 \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

(%i2) integrate(sqrt(1-x^2)*x, x,0,1);

(%o2) $\frac{1}{3}$

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函數為 $\sqrt{1-x^2}\,x$,區間範圍從 0 到 1 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example8 Find $\int_0^1 \frac{3x^2-1}{1+\sqrt{x-x^3}}dx.$

Let $u=x-x^3$. Then $du=(1-3x^2)dx$. When $x=0, u=0-0^3=0.$
When $x=1, u=1-1^3=0.$ Then

$$\int_0^1 \frac{3x^2-1}{1+\sqrt{x-x^3}}dx=\int_0^0 -\frac{du}{1+\sqrt{u}}=0.$$

As x goes from 0 to 1, u starts at 0, increases for a time, then drops back to 0.

(%i3) integrate((3*x^2-1)/(1+(sqrt(x-x^3))), x,0,1);

(%o3) $\int_0^1 \frac{3\,x^2-1}{\sqrt{x-x^3}+1}dx$

函數為 $\frac{3x^2-1}{1+\sqrt{x-x^3}}$,區間範圍從 0 到 1 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

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4.5 AREA BETWEEN TWO CURVES

Example1 Find the area of the region between the curves $y=\frac{1}{2}x^2-1$ and $y=x$ from $x=1$ to $x=2$. In Figure4.5.4, we sketch the curves to check that $\frac{1}{2}x^2-1\le x$ for $1\le x\le 2$. Then

$$A=\int_1^2 x-(\frac{1}{2}x^2-1)dx=\frac{1}{2}x^2-\frac{1}{6}x^3+x\Big]_1^2=\frac{8}{6.}$$

```
(%i7) integrate((x-(x^2/2-1)), x,1,2);
```

(%o7) $\frac{4}{3}$

函數爲 $x-(\frac{1}{2}x^2-1)$,區間範圍從 1 到 2 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example2 Find the area of the region bounded above by $y=x+2$ and below by $y=x^2$. Part of the problem is to find the limits of integration. First draw a sketch. The curves intersect at two points, which can be found by solving the equation $x+2=x^2$ for x.

$$x^2-(x+2)=0, \qquad (x+1)(x-2)=0,$$

$$x=-1 \quad \text{and} \quad x=2.$$

Then $$A=\int_{-1}^2 (x+2-x^2)dx=\frac{1}{2}x^2+2x-\frac{1}{3}x^3\Big]_{-1}^2=4\frac{1}{2.}$$

```
(%i9) integrate(x+2-x^2, x, -1, 2);
```

(%o9) $\frac{9}{2}$

函數爲 $x+2-x^2$,區間範圍從-1 到 2 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example 3 Find the area of the region R bounded below by the line $y = -1$ and above by the curves $y = x^3$ and $y = 2 - x$. The region is shown in Figure 4.5.6. This problem can be solved in three ways. Each solution illustrates a different tick which is useful in other area problems. The three corners of the region are:

(-1, -1), where $y = x^3$ and $y = -1$ cross.

(3, -1), where $y = 2 - x$ and $y = -1$ cross.

(1, 1), where $y = x^3$ and $y = 2 - x$ cross.

Note that $y = x^3$ and $y = 2 - x$ can cross at only one point because x^3 is Always increasing and $2 - x$ is always decreasing.

FIRST SOLUTION

Break the region into the two parts shown in Figure 4.5.7:

R_1 from $x = -1$ to $x = 1$, and R_2 from $x = 1$ to $x = 3$. Then

area of R = area of R_1 + area of R_2.

$$\text{area of } R_1 = \int_{-1}^{1} x^3 - (-1)\, dx = \frac{1}{4}x^4 + x \Big]_{-1}^{1} = 2.$$

$$\text{area of } R_2 = \int_{1}^{3} (2 - x) - (-1)\, dx = 3x - \frac{1}{2}x^2 \Big]_{1}^{3} = 2.$$

area of R =2+2=4.

SECOND SOLUTION

From the triangular region S between $y = -1$ and $y = 2 - x$ from -1 to 3. The region R is obtained by subtracting from S the region S_1 shown in Figure 4.5.8.

Then area of R = area of S - area of $S_{1.}$.

$$\text{area of } S = \int_{-1}^{3} (2 - x) - (-1)\, dx = 3x - \frac{1}{2}x^2 \Big]_{-1}^{3} = 8.$$

$$\text{area of } S_{1.} = \int_{-1}^{1} (2 - x) - x^3\, dx = 2x - \frac{1}{2}x^2 - \frac{1}{4}x^4 \Big]_{-1}^{1} = 4.$$

area of R =8-4=4.

THIRD SOLUTION

Use y as the independent variable and x as the dependent variable. Write the boundary curves with x as a function of y.

$y = 2 - x$ becomes $x = 2 - y$.

$y = x^3$ becomes $x = y^{1/3}$.

The limits of integration are $y = -1$ and $y = 1$ (see Figure 4.5.9). Then

$$A = \int_{-1}^{1} (2 - y) - y^{1/3} dy = 2y - \frac{1}{2}y^2 - \frac{3}{4}y^{4/3}\Big]_{-1}^{1} = 4.$$

As expected, all three solutions gave the same answer.

4.6 NUMERICAL INTEGRATION

Example1 Approximate the definite integral

$$\int_0^1 \sqrt{1+x^2}\, dx.$$

Use the trapezoidal approximation with $\Delta x = \frac{1}{5.}$ .We first make a table of values of.

$\sqrt{1+x^2}$. The graph is drawn in Figure 4.6.2.

x	$\sqrt{1+x^2}$	$\sqrt{1+x^2}$ to four places	term in trapezoidal approximation
$x_0 = 0$	1	1.0000	0.5000= $\frac{1}{2} f(x_0)$
$x_1 = \frac{1}{5}$	$\sqrt{1.04}$	1.0198	1.0198= $f(x_1)$
$x_2 = \frac{2}{5}$	$\sqrt{1.16}$	1.0770	1.0770= $f(x_1)$
$x_3 = \frac{3}{5}$	$\sqrt{1.36}$	1.1662	1.1662= $f(x_1)$
$x_4 = \frac{4}{5}$	$\sqrt{1.64}$	1.2806	1.2806= $f(x_1)$
$x_5 = 1$	$\sqrt{2}$	1.4142	0.7071= $f(x_1)$

Thus, $\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + \frac{1}{2} f(x_5) = 5.7507.$

Since $\Delta x = \frac{1}{5}$, the trapezoidal approximation is

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$$(5.7507)\cdot\frac{1}{5}=1.1501,$$

$$\int_0^1\sqrt{1+x^2}\,dx\sim 1.1501$$

(%i10) integrate(sqrt((1+x^2)), x, 0, 1);

(%o10) $\dfrac{\operatorname{asinh}(1)+\sqrt{2}}{2}$

函數為 $\sqrt{1+x^2}$,區間範圍從 0 到 1 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example2 Consider the integral

$$\int_{-1}^{1}\sqrt{1-x^2}\,dx=\pi/2.$$

Let $f(x)=\sqrt{1-x^2}.$

By Theorem1, we have

$$\lim_{\Delta x\to 0^+}\sum_{-1}^{1}\frac{f(x)+f(x+\Delta x)}{2}\Delta x=\frac{\pi}{2}.$$

However, the Trapezoidal Rule fails to give an error estimate in this case because $f'(x)$ is discontinuous at $x=\pm 1$.

(%i12) integrate(sqrt(1-x^2), x, -1, 1);

(%o12) $\dfrac{\pi}{2}$

函數為 $\sqrt{1-x^2}$,區間範圍從-1 到 1 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);

Example3 Use Simpson's Rule with $\Delta x = 0.25$ to approximate the integral

$$A = \int_0^1 e^{-x^2/2}\,dx$$

and find the error estimate.

The curve is the normal (bell-shaped) curve used in statistics, shown in Figure 4.6.5. We are to divide the interval [0, 1] into four subintervals of equal length $\Delta x = 0.25$. The following table shows the values of x and y and the coefficient to be used in Simpson's approximation for each partition point.

x	$e^{-x^2/2}$	Coefficient
0.0	1.000000	1
0.25	0.969233	4
0.5	0.882496	2
0.75	0.754840	4
1.0	0.606531	1

The sum used in the Simpson approximation is then

[1.000000+4(0.969233)+2(0.882496)+4(0.754840)+0.606531]=10.267816

To get the Simpson approximation, we multiply this sum by $\Delta x/3$:

$$S = (10.267816)\cdot(0.25)/3 = 0.855651.$$

To find the error estimate we need the fourth derivative of

$$y = e^{-x^2/2}.$$

The fourth derivative can be computed as usual and turns out to be

$$y^{(4)} == (x^4 - 6x^2 + 3)e^{-x^2/2}.$$

On the interval [0, 1], $y^{(4)}$ is decreasing because both $x^4 - 6x^2 + 3$ and $-x^2/2$ and decreasing, and therefore $y^{(4)}$ has its maximum value at $x = 0$ and its minimum value at $x := 1$,

maximum: $y^{(4)}(0) = 3$

minimum: $y^{(4)}(1) = -1.213061$

The maximum value of the absolute value $\left|y^{(4)}\right|$ is thus $M = 3$. The error estimate in Simpson's Rule is then

$$\frac{b-a}{180}(\Delta x)^4 M = \frac{1}{180}\cdot(0.25)^4\cdot 3 = 0.000065.$$

This shows that the integral is within 0.000065 of the approximation; that is,

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$$\int_0^1 e^{-x^2/2}dx = 0.855651 \pm 0.000065,$$

or using inequalities,

$$0.855586 \le \int_0^1 e^{-x^2/2}dx \le 0.855716.$$

For comparison, a more accurate computation with a smaller shows that the actual value to six places is

$$\int_0^1 e^{-x^2/2}dx = 0.855624.$$

The Trapezoidal Rule for this integral and the same value of $\Delta x = 0.25$ give an approximate value of 0.85246 for the integral and error estimate of 0.00521.

(%i13) integrate(e^((-x^2/2)), x, 0, 1);

(%o13) $\int_0^1 \frac{1}{e^{\frac{x^2}{2}}} dx$

函數爲 $e^{-x^2/2}$,區間範圍從 0 到 1 ,可使用指令

integrate(函數, 變數, 範圍最小值, 範圍最大值);