Maxima 在微積分上之應用

微分的應用

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3.1 Extrema on an Interval

Example 1.　The Value of the Derivative at Relative Extrema

Find the value of the derivative at each of the relative extrema shown in Figure 3.3.

Solution：

a. (%i1) f:(9*(x^2-3))/(x^3);　//定義一函數 $\frac{9(x^2-3)}{x^3}$ ，函數名稱叫做 f

(%o1) $\frac{9(x^2-3)}{x^3}$

(%i2) plot2d([f],[x,0,7],[y,-4,3]);　繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $\frac{9(x^2-3)}{x^3}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-4~3，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

// Relative maximum 出現在(3, 2)

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plot2d: expression evaluates to non-numeric value somewhere in plot
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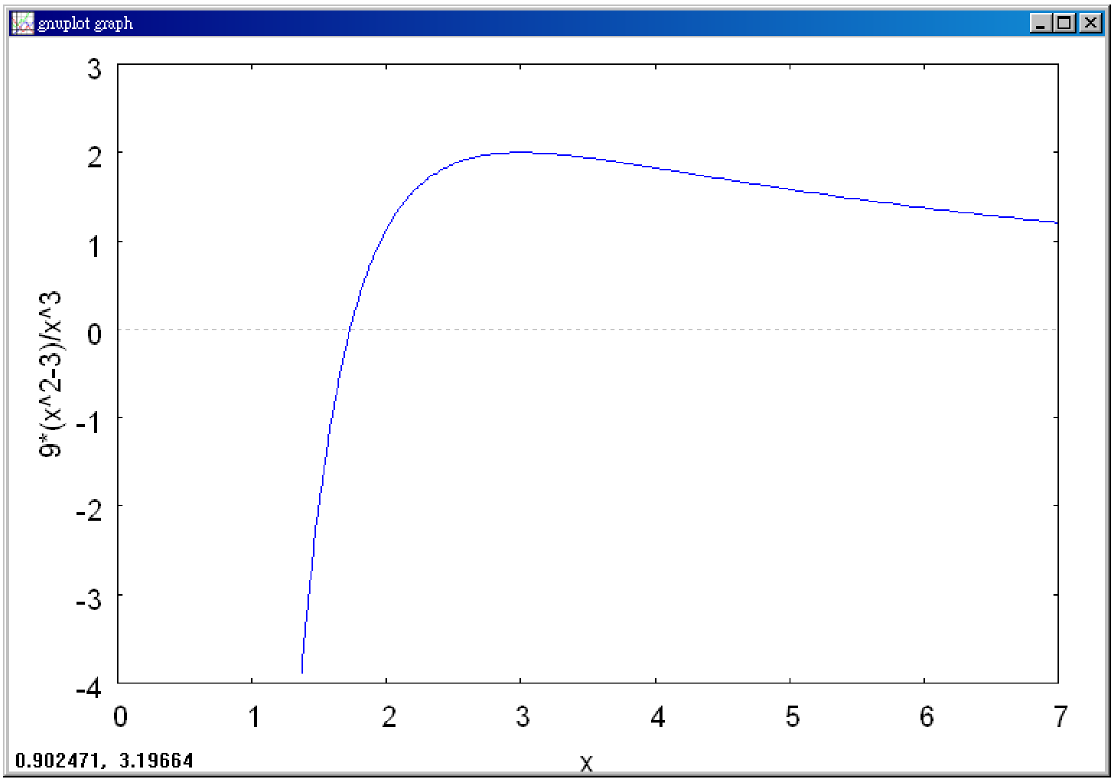
(%o2)



(%i3) f:(9*(x^2-3))/(x^3); //建立一函數 $\frac{9(x^2-3)}{x^3}$ ，方程式名稱叫做 f

(%o3) $\frac{9(x^2-3)}{x^3}$

(%i4) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o4) $\frac{18}{x^2}-\frac{27(x^2-3)}{x^4}$

(%i5) g(x):=(18/x^2)-(27*(x^2-3)/x^4); //將微分後的函數名稱給定為 g(x)

(%o5) $g(x):=\frac{18}{x^2}-\frac{27(x^2-3)}{x^4}$

(%i6) g(3); //將 x=3 代入函數 g(x)中→可得知 $f'(3)=0$

(%o6) 0

The derivative of $f(x) = \frac{9(x^2 - 3)}{x^3}$ is

$$f'(x) = \frac{x^3(18x) - (9)(x^2 - 3)(3x^2)}{(x^3)^2}$$ Differentiate using Quotient Rule.

$$= \frac{9(9 - x^2)}{x^4}.$$ Simplify.

At the point (3, 2), the value of the derivative is $f'(3) = 0$ [see Figure 3.3(a)].

b. (%i1) f:abs(x); //定義一函數$|x|$，函數名稱叫做 f

(%o1) $|x|$

(%i2) plot2d([f],[x,-2,2],[y,-2,3]); 繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

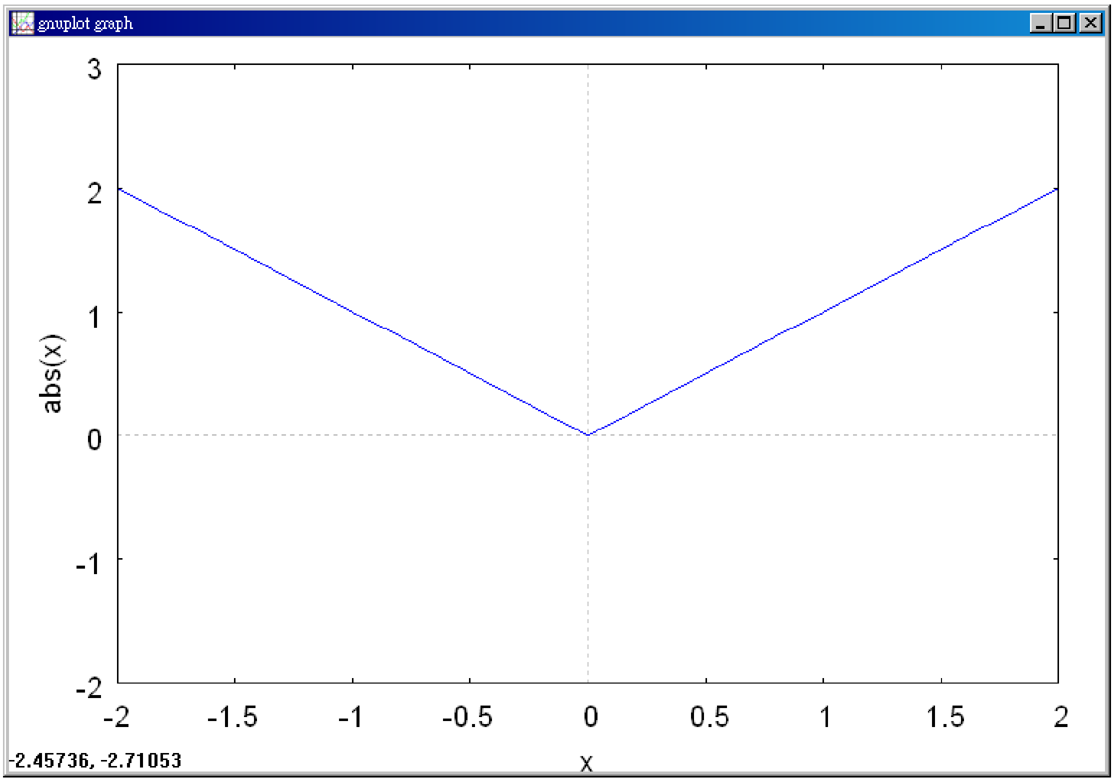
expr：是你要繪製的函數，這例是$|x|$函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-2~3，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

// Relative maximum 出現在(0, 0)

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o3) $\frac{x}{|x|}$

(%i4) g(x):=x/abs(x); //將微分後的函數名稱給定爲 g(x)

(%o4) $g(x):=\frac{x}{|x|}$

(%i5) g(0); //將 x=3 代入函數 g(x)中→可得知 $f'(0)$ 不存在

```
Division by 0
#0: g(x=0)
 -- an error.  To debug this try debugmode(true);
```

(%i6) 'limit(g(x),x,0,plus); 有時我們會希望呈現未運算前的格式，在 maxima，我們只需要在方程式前面加上「 ' 」，maxima 就不會運算該程式

$$(\%o6) \quad \lim_{x \to 0+} \frac{x}{|x|}$$

(%i7) limit(g(x),x,0,plus); //求函數 g(x)對 x=0$^+$微分，plus 指的是右極限

(%i8) 'limit(g(x),x,0,minus); 有時我們會希望呈現未運算前的格式，在 maxima，我們只需要在方程式前面加上「 ' 」，maxima 就不會運算該程式

$$(\%o8) \quad \lim_{x \to 0-} \frac{x}{|x|}$$

(%i9) limit(g(x),x,0,minus); //求函數 g(x)對 x=0$^-$微分，minus 指的是左極限

(%o9) -1

(%i10) limit(g(x),x,0); 求極限若結果出現 und 代表極限不存在 //求 g(x)在 x=0 的極限

(%o6) *und*

At $x = 0$, the derivative of $f(x) = |x|$ does not exist because the following one-sided limits differ [see Figure 3.3(b)].

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = -1 \qquad \text{Limit from the left}$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = 1 \qquad \text{Limit from the right}$$

c. (%i1) f:sin(x); //定義一函數 $\sin x$ ，函數名稱叫做 f

(%o1) sin(x)

(%i2) plot2d([f],[x,0,2*(%pi)],[y,-2,2]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

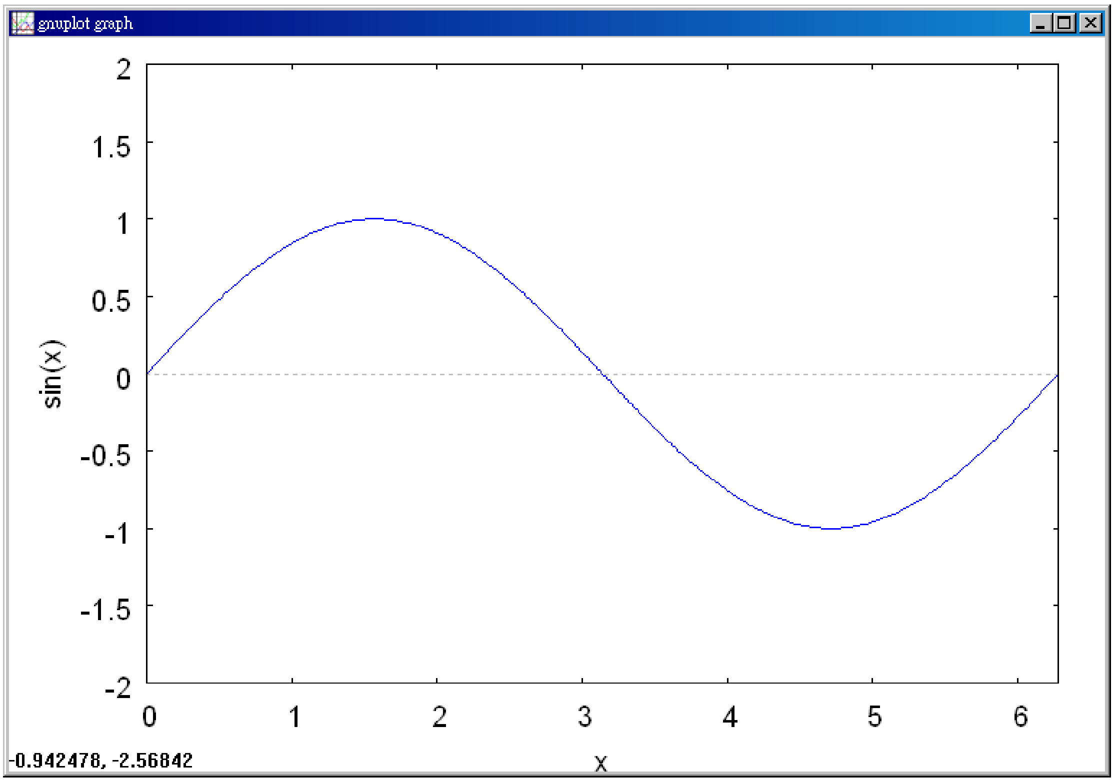
expr：是你要繪製的函數，這例是 $\sin x$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-2~2，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‥‥等等。

// Relative maximum 出現在$(\frac{\pi}{2},1)$而 relative minimum 出現在$(\frac{3\pi}{2},-1)$

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o3) cos(x)

(%i4) g(x):=cos(x);　//將微分後的函數名稱給定爲 g(x)

(%o4)　g(x):=cos(x)

(%i5) g(%pi/2);　//將 x=$\frac{\pi}{2}$代入函數 g(x)中→可得知 $f'(\frac{\pi}{2})=0$

(%o5)　0

(%i6) g(3*(%pi)/2);　//將 x=$\frac{3\pi}{2}$代入函數 g(x)中→可得知 $f'(\frac{3\pi}{2})=0$

(%o6)　0

The derivative of $f(x)=\sin x$ is $f'(x)=\cos x$.

At the point ($\frac{\pi}{2}$, 1), the value of the derivative is $f'(\frac{\pi}{2})=\cos(\frac{\pi}{2})=0$. At the point ($\frac{3\pi}{2}$,−1), the value of the derivative is $f'(\frac{3\pi}{2})=\cos(\frac{3\pi}{2})=0$ [see Figure 3.3(c)].

Example 2.　Finding Extrema on a Closed Interval

Find the extrema of $f(x)=3x^4-4x^3$ on the interval [-1, 2].

Solution：(%i1) f(x):=3*x^4-4*x^3;　//定義一函數$3x^4-4x^3$，函數名稱叫做 f

(%o1)　f(x):=3 x^4 - 4 x^3

(%i2) plot2d([f],[x,-1,2]);　繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是$3x^4-4x^3$函數圖形

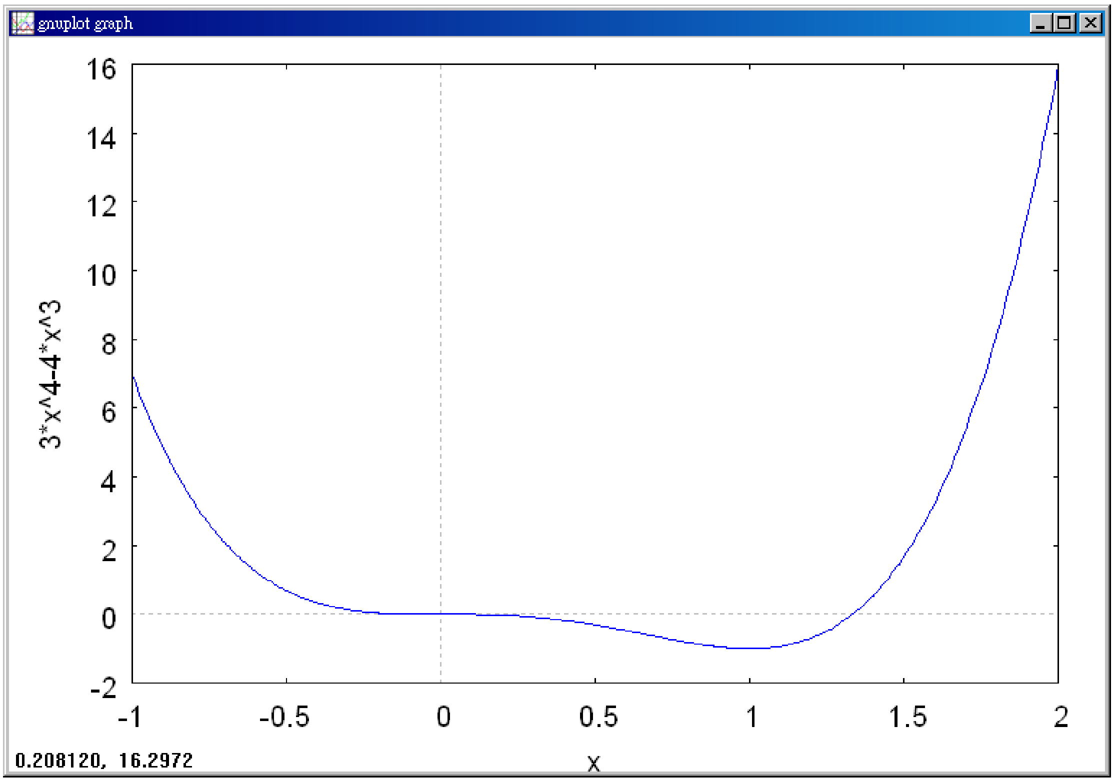
x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-1~2，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等

等。

(%o2)

// On the closed interval [-1, 2]，f has a minimum at (1, -1) and a maximum at (2, 16)



(%i3) diff(f(x),x); 微分的指令：differ(函數，要微分的變數) //對函數 f(x)中的 x 變數微分

(%o3) $12\ x^3 - 12\ x^2$

(%i4) f(-1); //將 x=-1 代入 f(x)求得值 7

(%o4) 7

(%i5) f(0); //將 x=0 代入 f(x)求得值 0

(%o5) 0

(%i6) f(1); //將 x=1 代入 f(x)求得值-1

(%o6) - 1

(%i7) f(2);　//將 x=2 代入 f(x)求得值 16

(%o7) 16

Begin by differentiating the function.

$f(x) = 3x^4 - 4x^3$　　Write original function.

$f'(x) = 12x^3 - 12x^2$　　Differentiate.

To find the critical numbers of f, you must find all x-values for which $f'(x) = 0$ and all x-values for which $f'(x)$ does not exist.

$f'(x) = 12x^3 - 12x^2 = 0$　　Set $f'(x)$ equal to 0.

$12x^2(x-1) = 0$　　Factor.

$x = 0, \quad 1$　　Critical numbers

Because f' is defined for all x, you can conclude that these are the only critical numbers of f. By evaluating f at these two critical numbers and the endpoints of [-1, 2], you can determine that the maximum is $f(2) = 16$ and the minimum is $f(1) = -1$, as shown in the table. The graph of f is shown in Figure 3.5.

Left Endpoint	**Critical Number**	**Critical Number**	**Right Endpoint**
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum

Example 3.　Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on the interval [-1, 3].

Solution：(%i1) f(x):=2*x-3*x^(2/3);　//定義一函數 $2x - 3x^{2/3}$，函數名稱叫做 f

(%o1) $f(x) := 2\,x - 3\,x^{2/3}$

(%i2) plot2d([f(x)],[x,-2,3],[y,-6,1]);　繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $2x-3x^{2/3}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-6~1，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

//On the closed interval [-1, 3], f has a minimum at (-1, -5) and a maximum at (0, 0)

```
plot2d: some values were clipped.
 (%o2)
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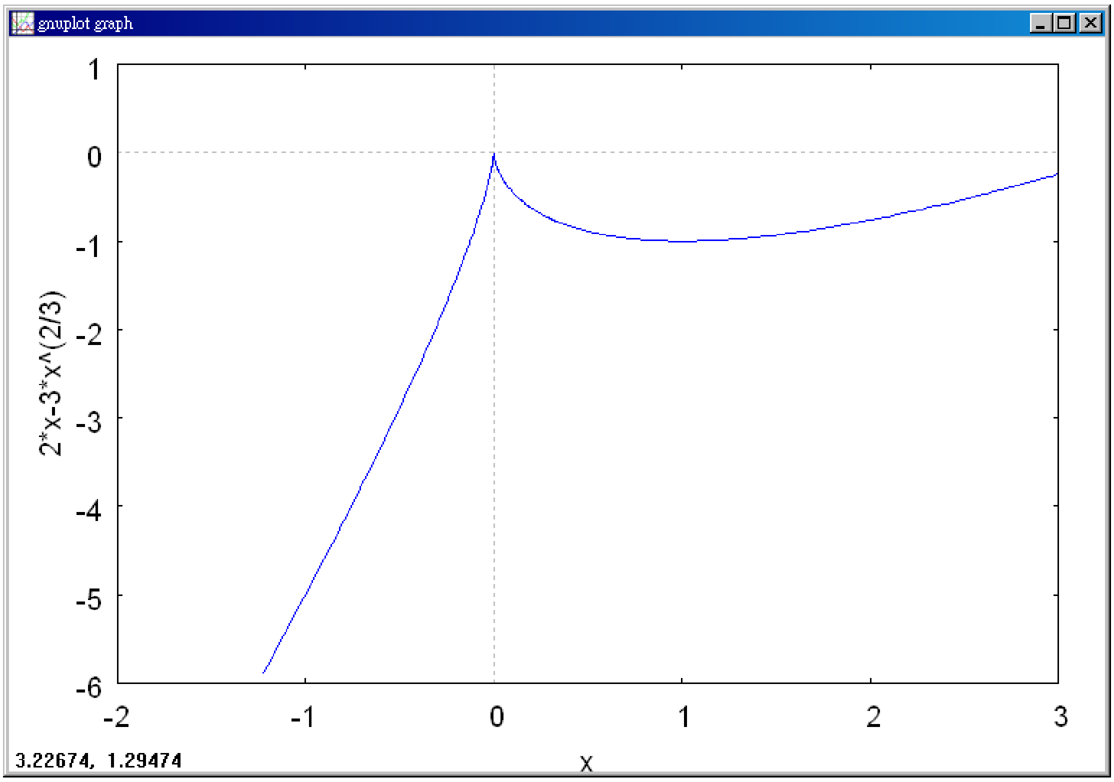

(%i3) diff(f(x),x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o3) $2-\dfrac{2}{x^{1/3}}$

(%i4) f(-1); //將 x=-1 代入 f(x)求得值-5

(%o4) -5

(%i5) f(0); //將 x=0 代入 f(x)求得值 0

(%o5) 0

(%i6) f(1); //將 x=1 代入 f(x)求得值-1

(%o6) -1

(%i7) f(3); //將 x=3 代入 f(x)求得值 $6-3^{5/3}$

(%o7) $6-3^{5/3}$

Begin by differentiating the function.

$f(x)=2x-3x^{2/3}$ Write original function.

$f'(x)=2-\frac{2}{x^{1/3}}=2(\frac{x^{1/3}-1}{x^{1/3}})$ Differentiate.

From this derivative, you can see that the function has two critical numbers in the interval [-1, 3]. The number 1 is a critical number because $f'(1)=0$, and the number 0 is a critical number because $f'(0)$ does not exist. By evaluating f at these two numbers and at the endpoints of the interval, you can conclude that the minimum is $f(-1)=-5$ and the maximum is $f(0)=0$, as shown in the table. The graph of f is shown in Figure 3.6.

Left Endpoint	Critical Number	Critical Number	Right Endpoint
$f(-1)=-5$ Minimum	$f(0)=0$ Maximum	$f(1)=-1$	$f(3)=6-3\sqrt[3]{9}\approx -0.24$

Example 4. Finding Extrema on a Closed Interval

Find the extrema of $f(x)=2\sin x-\cos 2x$ on the interval [0, 2π].

Solution：(%i1) f(x):=2*sin(x)-cos(2*x);　//定義一函數 $2\sin x - \cos 2x$ ，函數名稱叫做 f

```
(%o1) f(x):=2 sin(x)-cos(2 x)
```

(%i2) plot2d([f(x)],[x,0,2*%pi]); 繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $2\sin x - \cos 2x$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

//On the closed interval [0, 2π], f has two minimum at (7π/6, -3/2) and (11π/6, -3/2) and a maximum at (π/2, 3)

```
(%o2)
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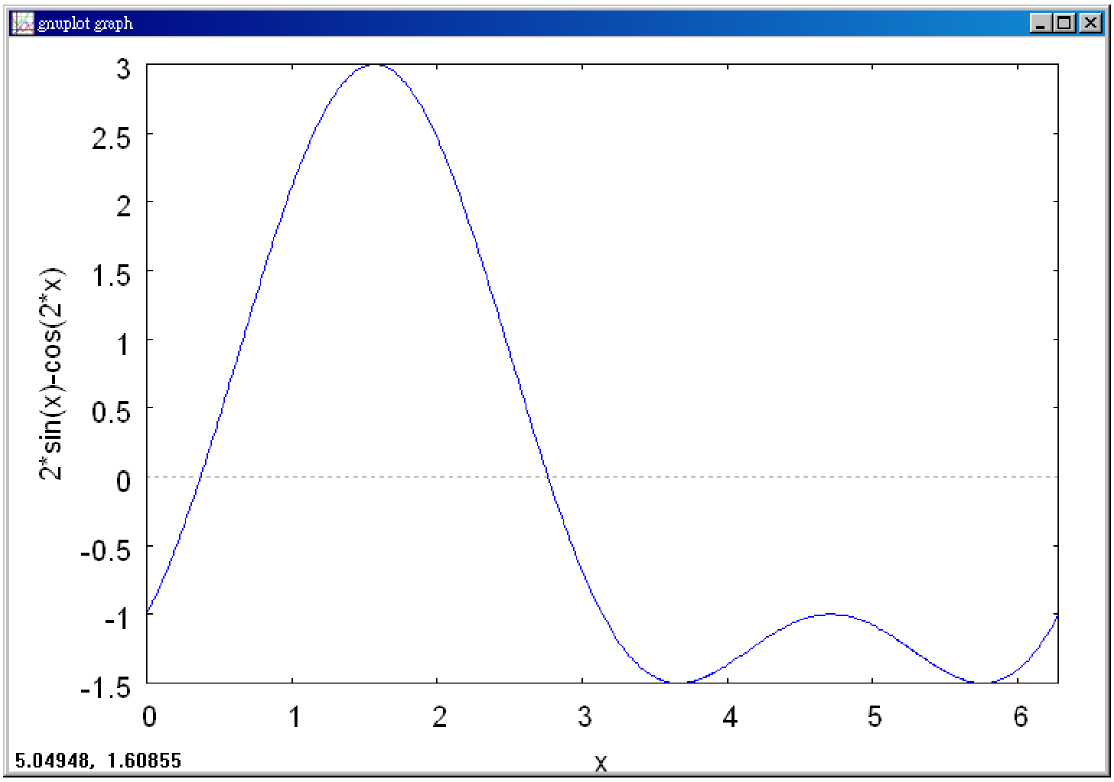


(%i3) diff(f(x),x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o3) 2 sin(2 x)+2 cos(x)

(%i4) f(0); //將 x=0 代入 f(x)求得值-1

(%o4) −1

(%i5) f(%pi/2); //將 $x=\frac{\pi}{2}$ 代入 f(x)求得值 3

(%o5) 3

(%i6) f(7*%pi/6); //將 $x=\frac{7\pi}{6}$ 代入 f(x)求得值$-\frac{3}{2}$

(%o6) $-\frac{3}{2}$

(%i7) f(3*%pi/2); //將 x=$\frac{3\pi}{2}$ 代入 f(x)求得值-1

(%o7) -1

(%i8) f(11*%pi/6); //將 x=$\frac{11\pi}{6}$ 代入 f(x)求得值$-\frac{3}{2}$

(%o8) $-\frac{3}{2}$

(%i9) f(2*%pi); //將 x=2π 代入 f(x)求得值-1

(%o9) -1

This function is differentiable for all real x, so you can find all critical numbers by differentiating the function and setting $f'(x)$ equal to zero, as shown.

$f(x) = 2\sin x - \cos 2x$	Write original function.
$f'(x) = 2\cos x + 2\sin 2x = 0$	Set $f'(x)$ equal to 0.
$2\cos x + 4\cos x \sin x = 0$	sin2x=2cosxsinx
$2(\cos x)(1+2\sin x) = 0$	Factor.

In the interval [0, 2π], the factor $\cos x$ is zero when $x = \pi/2$ and when $x = 3\pi/2$. The factor (1+2$\sin x$) is zero when $x = 7\pi/6$ and when $x = 11\pi/6$. By evaluating f at these four critical numbers and at the endpoints of the interval, you can conclude that the maximum is $f(\pi/2) = 3$ and the minimum occurs at two points, $f(7\pi/6) = -3/2$ and $f(11\pi/6) = -3/2$, as shown in the table. The graph is shown in Figure 3.7.

Left Endpoint	Critical Number	Critical Number	Critical number	Critical number	Right Endpoint
$f(0) = -1$	$f(\frac{\pi}{2}) = 3$ Maximum	$f(\frac{7\pi}{6}) = -\frac{3}{2}$ Minimum	$f(\frac{3\pi}{2}) = -1$	$f(\frac{11\pi}{6}) = -\frac{3}{2}$ Minimum	$f(2\pi) = -1$

3.2 Rolle's Theorem and the Mean Value Theorem

Example 1.　Illustrating Rolle's Theorem

Find the two x-intercepts of $f(x)=x^2-3x+2$ and show that $f'(x)=0$ at some point between the two x-intercepts.

Solution：(%i1) f:x^2-3*x+2;　//定義一函數 x^2-3x+2 ，函數名稱叫做 f

(%o1) x^2-3x+2

(%i2) plot2d([f],[x,-1,4]);　繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

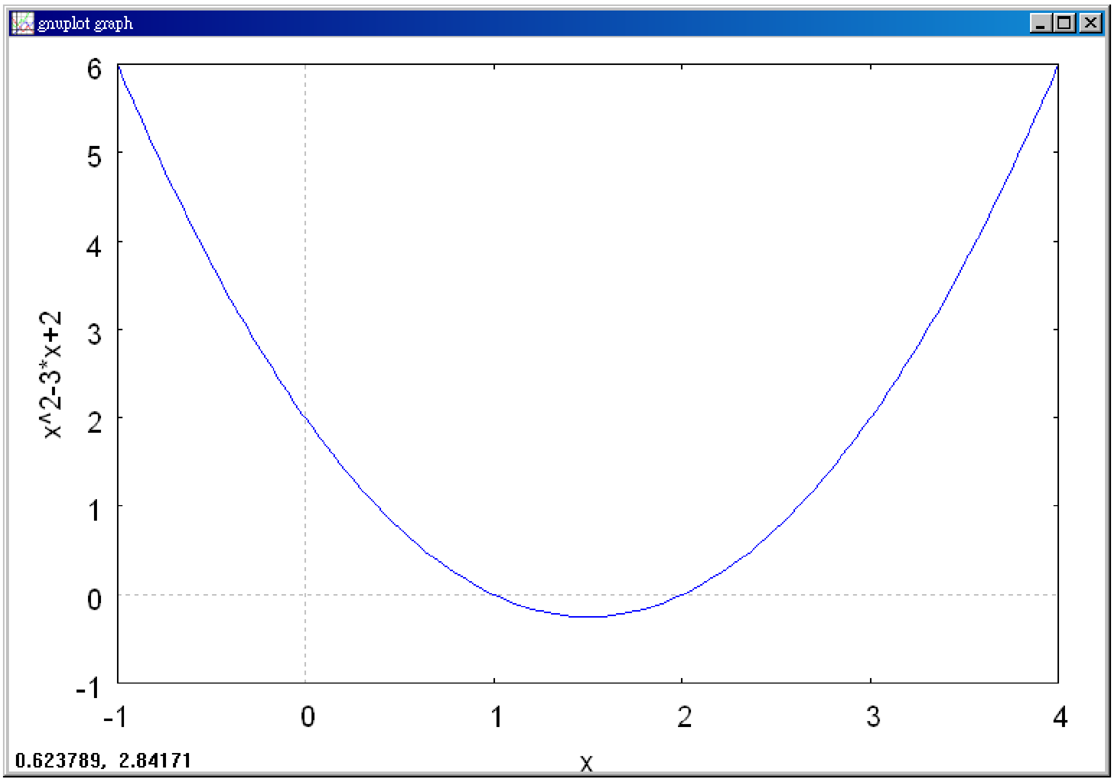
expr：是你要繪製的函數，這例是 x^2-3x+2 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

// The x-value for which $f'(x)=0$ is between the two x-intercepts.

(%o2)



(%i3) diff(f,x);　微分的指令：differ(函數，要微分的變數)　//對函數 f 中的 x 變數微分

(%o3) 2 x−3

Note that f is differentiable on the entire real line. Setting $f(x)$ equal to 0 produces $x^2 - 3x + 2 = 0$　　Set $f(x)$ equal to 0.

$(x-1)(x-2) = 0$.　　Factor.

So, $f(x) = f(2) = 0$, and from Rolle's Theorem you know that there exists at least one c in the interval (1, 2) such that $f'(c) = 0$. To find such a c, you can solve the equation $f'(x) = 2x - 3 = 0$　　Set $f'(x)$ equal to 0.

and determine that $f'(x) = 0$ when $x = \frac{3}{2}$. Note that the x-value lies in the open interval (1, 2), as shown in Figure 3.9.

Example 2.　Illustrating Rolle's Theorem

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval (-2, 2) such that $f'(c) = 0$.

Solution：(%i1) f:x^4-2*x^2; //定義一函數 $x^4 - 2x^2$ ，函數名稱叫做 f

(%o1) $x^4 - 2x^2$

(%i2) plot2d([f],[x,-2,2]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

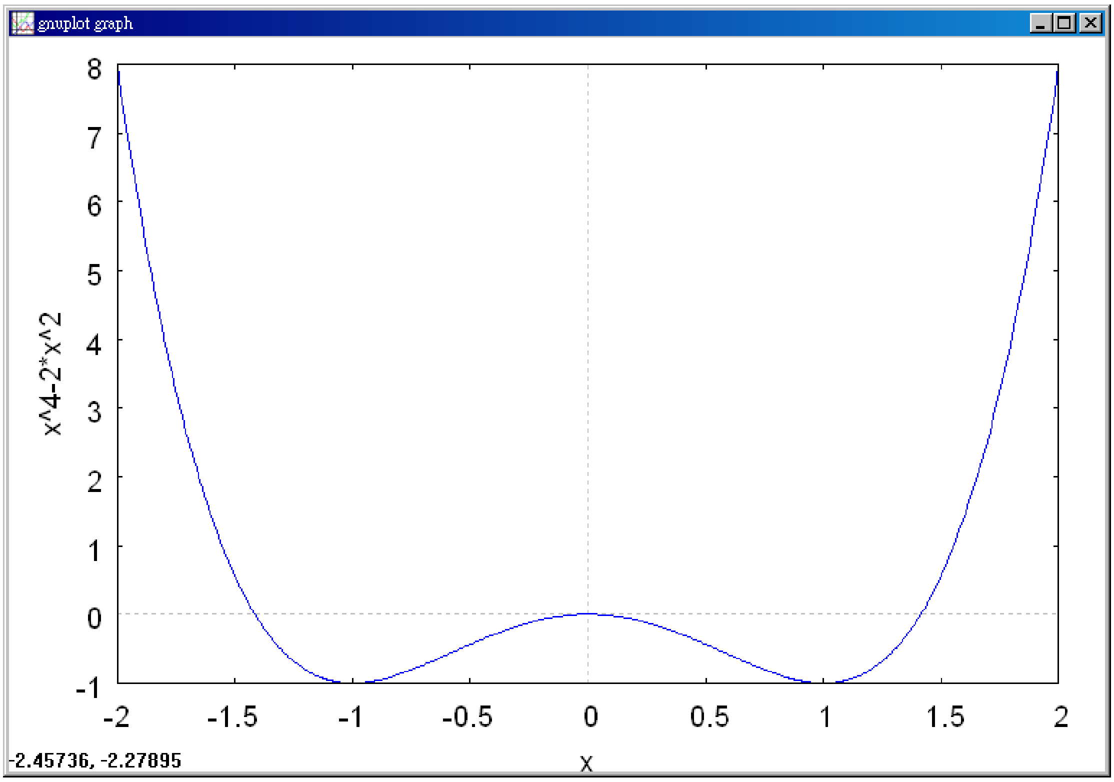
expr：是你要繪製的函數，這例是 $x^4 - 2x^2$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

// $f'(x) = 0$ for more than one x-value in the interval (-2, 2).

(%o2)



(%i3) diff(f,x);　微分的指令：differ(函數，要微分的變數)　//對函數 f 中的 x 變數微分

(%o3) $4x^3-4x$

To begin, note that the function satisfies the conditions of Rolle's Theorem. That is, f is continuous on the interval (-2, 2). Moreover, because $f(-2)=f(2)=8$, you can conclude that there exists at least one c in (-2, 2) such that $f'(c)=0$. Setting the derivative equal to 0 produces

$f'(x)=4x^3-4x=0$　　Set $f'(x)$ equal to 0.

$4x(x-1)(x+1)=0$　　Factor.

$x=0,1,-1$　　x -values for which $f'(x)=0$

So, in the interval (-2, 2), the derivative is zero at three different values of x, as shown in Figure 3.10.

Example 3.　Finding a Tangent Line

Given $f(x)=5-(4/x)$, find all values of c in the open interval (1, 4) such that $f'(c)=\dfrac{f(4)-f(1)}{4-1}$.

Solution：(%i1) f:5-(4/x);　//定義一函數 $5-(4/x)$，函數名稱叫做 f

(%o1) $5-\dfrac{4}{x}$

(%i2) plot2d([f],[x,1,4]);　繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

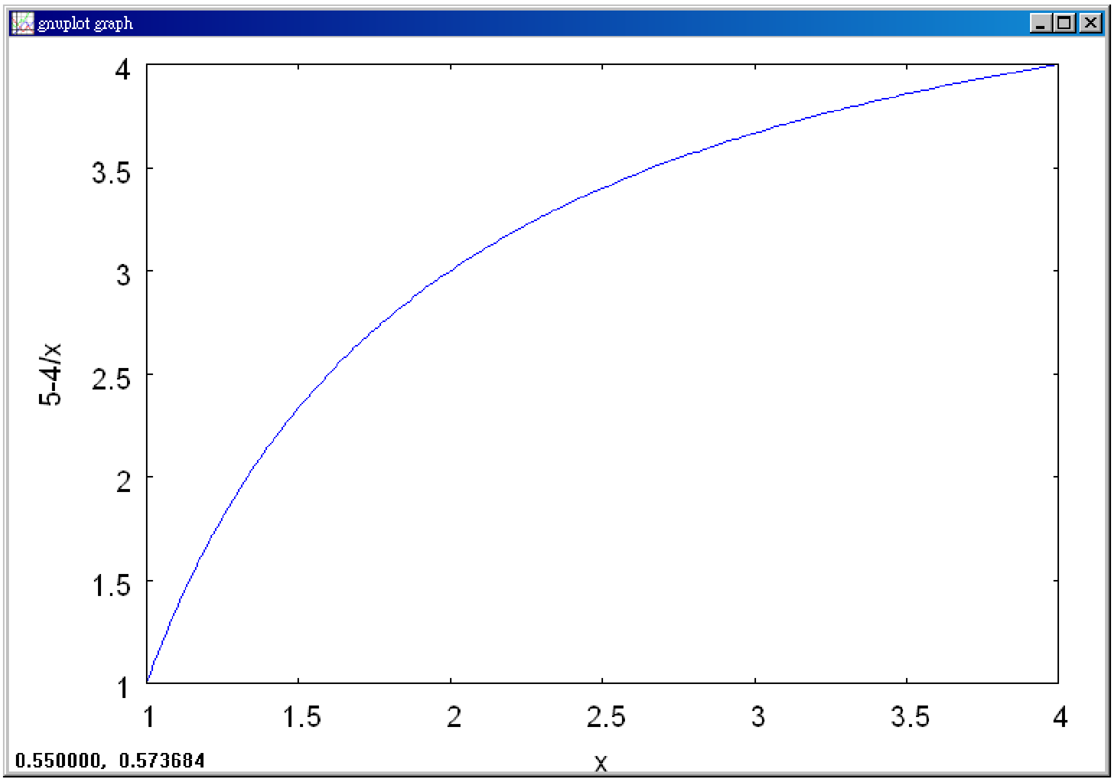
expr：是你要繪製的函數，這例是 $5-(4/x)$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

// The tangent line at (2, 3) is parallel to the secant line through (1, 1) and (4, 4).

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o3) $\frac{4}{x^2}$

The slope of the secant line through $(1, f(1))$ and $(4, f(4))$ is

$$\frac{f(4)-f(1)}{4-1}=\frac{4-1}{4-1}=1.$$

Because f satisfies the conditions of the Mean Value Theorem, there exists at least one number c in (1, 4) such that $f'(c)=1$. Solving the equation $f'(x)=1$ yields

$$f'(x)=\frac{4}{x^2}=1$$

which implies that $x=\pm 2$. So, in the interval (1, 4), you can conclude that $c=2$, as shown in Figure 3.13.

Example 4. Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 3.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) At some time during the 4 minutes.

Solution：

Let $t=0$ be the time (in hours) when the truck passes the first patrol car. The time when the truck passes the second patrol car is $t=\frac{4}{60}=\frac{1}{15}$ hour.

By letting $s(t)$ represent the distance (in miles) traveled by the truck, you have $s(0)=0$ and $s(\frac{1}{15})=5$. So, the average velocity of the truck over the five-mile stretch of highway is

$$\text{Average velocity}=\frac{s(1/15)-s(0)}{(1/15)-0}$$

$$=\frac{5}{1/15}=75 \text{ miles per hour.}$$

Assuming that the position function is differentiable, you can apply the Mean Value Theorem to conclude that the truck must have been traveling at a rate of 75 miles per hour sometime during the 4 minutes.

3.3 Increasing and Decreasing functions and the First Derivative Test

Example 1. Intervals on Which f Is Increasing or Decreasing

Find the open intervals on which $f(x)=x^3-\frac{3}{2}x^2$ is increasing or decreasing.

Solution：(%i1) f:x^3-(3/2)*x^2; //定義一函數 $x^3-\frac{3}{2}x^2$ ，函數名稱叫做 f

(%o1) $x^3-\frac{3\,x^2}{2}$

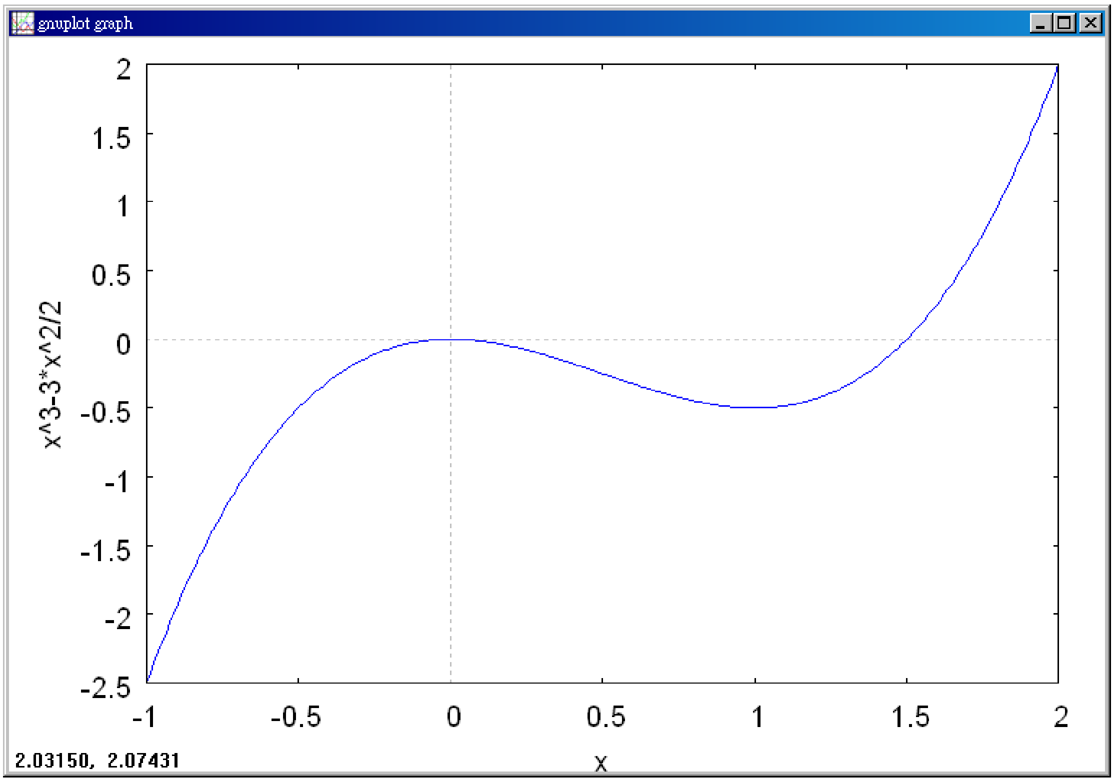
(%i2) plot2d([f],[x,-1,2]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $x^3-\frac{3}{2}x^2$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o3) $3\ x^2 - 3\ x$

(%i4) g(x):=3*x^2-3*x; //將微分後的函數名稱給定為 g(x)

(%o4) $g(x) := 3\ x^2 - 3\ x$

(%i5) g(-1); //將 x=-1 代入函數 g(x)中→可得知 $f'(-1) = 6$

(%o5) 6

(%i6) g(1/2); //將 x=1/2 代入函數 g(x)中→可得知 $f'(1/2) = -3/4$

(%o6) $-\frac{3}{4}$

(%i7) g(2); //將 x=2 代入函數 g(x)中→可得知 $f'(2) = 6$

(%o7) 6

Note that f is differentiable on the entire real number line. To determine the critical numbers of f, set $f'(x)$ equal zero.

$f(x) = x^3 - \frac{3}{2}x^2$ Write original function.

$f'(x) = 3x^2 - 3x = 0$ .Differentiate and set $f'(x)$ equal to 0.

$3(x)(x-1) = 0$ Factor.

$x = 0,1$ Critical numbers.

Because there are no points for which f' does not exist, you can conclude that $x = 0$ and $x = 1$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-1) = 6 > 0$	$f'(\frac{1}{2}) = -\frac{3}{4} < 0$	$f'(2) = 6 > 0$
Conclusion	Increasing	Decreasing	Increasing

So, f is increasing on the intervals (-∞, 0) and (1, ∞) and decreasing on the interval (0, 1), as shown in Figure 3.16.

Example 2. Applying the First Derivative Test

Find the relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ in the interval (0, 2π).

Solution：(%i1) f:(1/2)*x-sin(x); //定義一函數 $\frac{1}{2}x - \sin x$ ，函數名稱叫做 f

(%o1) $\frac{x}{2} - \sin(x)$

(%i2) plot2d([f],[x,0,2*(%pi)]); 繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

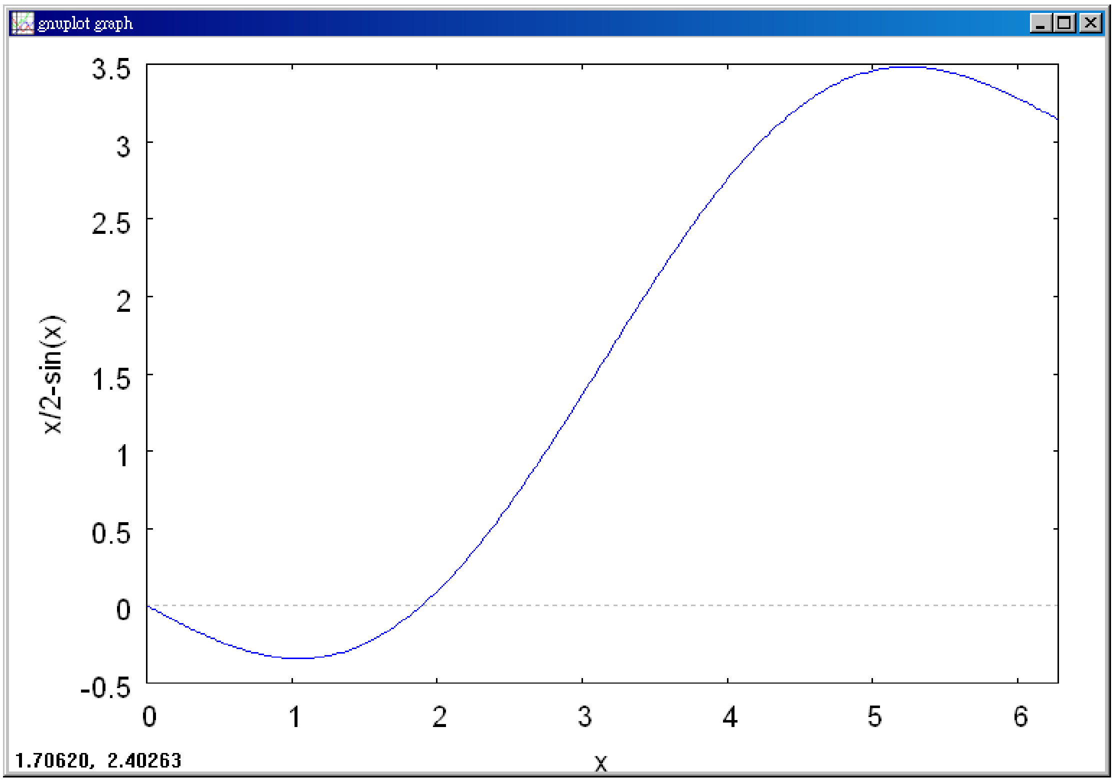
expr：是你要繪製的函數，這例是 $\frac{1}{2}x - \sin x$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

//A relative minimum occurs where f changes from decreasing to increasing, and a relative maximum occurs where f changes from increasing to decreasing.

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o3) $\frac{1}{2}-\cos(x)$

(%i4) g(x):=1/2-cos(x); //將微分後的函數名稱給定爲 g(x)

(%o4) $g(x):=\frac{1}{2}-\cos(x)$

(%i5) g(%pi/4); //將 x=$\frac{\pi}{4}$代入函數 g(x)中→可得知 $f'(\frac{\pi}{4})=\frac{1}{2}-\frac{1}{\sqrt{2}}$

(%o5) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

(%i6) g(%pi); //將 x=π代入函數 g(x)中→可得知 $f'(\pi)=\frac{3}{2}$

(%o6) $\frac{3}{2}$

(%i7) g(7*%pi/4); //將 x=$\frac{7\pi}{4}$代入函數 g(x)中→可得知 $f'(\frac{7\pi}{4})=\frac{1}{2}-\frac{1}{\sqrt{2}}$

(%o7) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

Note that f is continuous on the interval (0, 2π). To determine the critical numbers of f in this interval, set $f'(x)$ equal to 0.

$f'(x)=\frac{1}{2}-\cos x=0$ Set $f'(x)$ equal to 0.

$\cos x=\frac{1}{2}$

$x=\frac{\pi}{3},\frac{5\pi}{3}$ Critical numbers

Because there are no points for which f' does not exist, you can conclude that $x=\pi/3$ and $x=5\pi/3$ are the only critical numbers. The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test Value	$x = \frac{\pi}{4}$	$x = \pi$	$x = \frac{7\pi}{4}$
Sign of $f'(x)$	$f'(\frac{\pi}{4}) < 0$	$f'(\pi) > 0$	$f'(\frac{7\pi}{4}) < 0$
Conclusion	Decreasing	Increasing	Decreasing

By applying the First Derivative Test, you can conclude that f has a relative minimum at the point where

$x = \frac{\pi}{3}$ x -value where relative minimum occurs

and a relative maximum at the point where

$x = \frac{5\pi}{3}$ x -value where relative maximum occurs

as shown in Figure 3.19.

Example 3. Applying the First Derivative Test

Find the relative extrema of $f(x) = (x^2 - 4)^{2/3}$.

Solution：(%i1) f:(x^2-4)^(2/3); //定義一函數 $(x^2 - 4)^{2/3}$ ，函數名稱叫做 f

(%o1) $(x^2 - 4)^{2/3}$

(%i2) plot2d([f],[x,-4,4]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

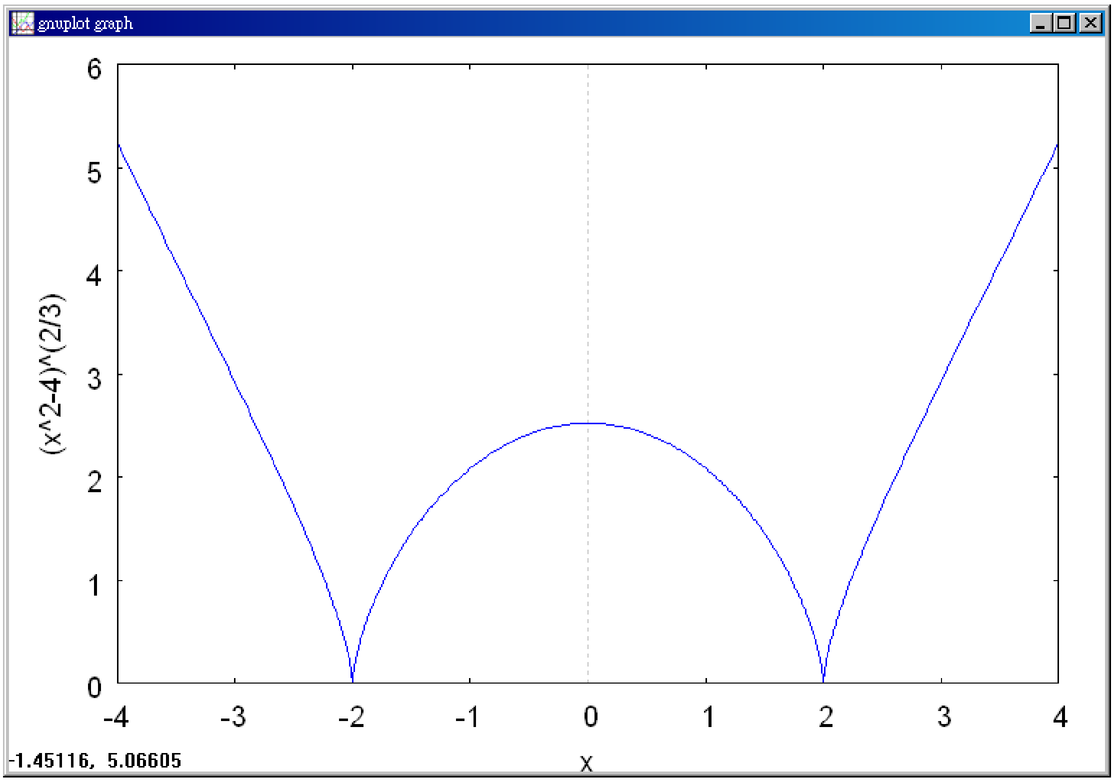
expr：是你要繪製的函數，這例是 $(x^2 - 4)^{2/3}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

//You can apply the First Derivative Test to find relative extrema.

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o3) $\dfrac{4\,x}{3\,(x^2-4)^{1/3}}$

(%i4) g(x):=(4*x)/(3*(x^2-4))^(1/3); //將微分後的函數名稱給定為 g(x)

(%o4) $g(x):=\dfrac{4\,x}{(3\,(x^2-4))^{1/3}}$

(%i5) g(-3); //將 x=-3 代入函數 g(x)中→可得知 $f'(-3)<0$

(%o5) $-\dfrac{12}{15^{1/3}}$

(%i6) g(-1); //將 x=-1 代入函數 g(x)中→可得知 $f'(-1)>0$

(%o6) $\dfrac{4}{9^{1/3}}$

(%i7) g(1); //將 x=1 代入函數 g(x)中→可得知 $f'(1) < 0$

(%o7) $-\frac{4}{9^{1/3}}$

(%i8) g(3); //將 x=3 代入函數 g(x)中→可得知 $f'(3) > 0$

(%o8) $\frac{12}{15^{1/3}}$

Begin by noting that f is continuous on the entire real line. The derivative of f

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) \qquad \text{General Power Rule}$$

$$= \frac{4x}{3(x^2 - 4)^{1/3}} \qquad \text{Simplify.}$$

is 0 when $x = 0$ and does not exist when $x = \pm 2$. So, the critical numbers are $x = -2, x = 0$, and $x = 2$. The table summarizes the testing of the four intervals determined by these three critical numbers.

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = -1$	$x = 1$	$x = 3$
Sign of $f'(x)$	$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

By applying the First Derivative Test, you can conclude that f has relative minimum at the point (-2, 0) , a relative maximum at the point (0, $\sqrt[3]{16}$), and another relative minimum at the point (2, 0), as shown in Figure 3.20.

Example 4. Applying the first Derivative Test

Find the relative extrema of $f(x) = \frac{x^4 + 1}{x^2}$.

Solution：(%i1) f:(x^4+1)/(x^2); //定義一函數$\frac{x^4 + 1}{x^2}$，函數名稱叫做 f

(%o1) $\frac{x^4+1}{x^2}$

(%i2) plot2d([f],[x,-3,3],[y,0,5]); 繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

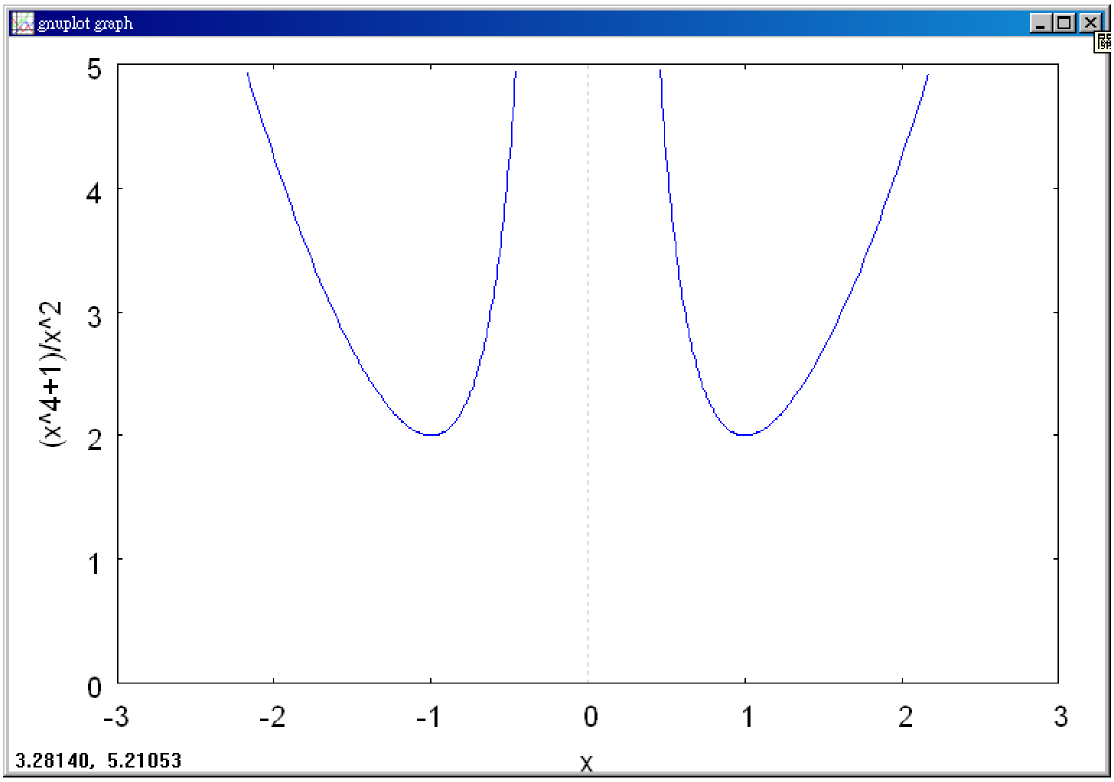
expr：是你要繪製的函數，這例是$\frac{x^4+1}{x^2}$函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍 0~5，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

// x-values that are not in the domain of f, as well as critical numbers, determine test intervals for f'.

```
plot2d: expression evaluates to non-numeric value somewhere in plott
plot2d: some values were clipped.
  (%o2)
```



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

$$(\%o3) \quad 4x-\frac{2(x^4+1)}{x^3}$$

(%i4) g(x):=4*x-(2*(x^4+1)/(x^3)); //將微分後的函數名稱給定為 g(x)

$$(\%o4) \quad g(x):=4x-\frac{2(x^4+1)}{x^3}$$

(%i5) g(-2); //將 x=-2 代入函數 g(x)中→可得知 $f'(-2)<0$

$$(\%o5) \quad -\frac{15}{4}$$

(%i6) g(-1/2); //將 x=-1/2 代入函數 g(x)中→可得知 $f'(-\frac{1}{2})>0$

$$(\%o6) \quad 15$$

(%i7) g(1/2); //將 x=1/2 代入函數 g(x)中→可得知 $f'(\frac{1}{2})<0$

$$(\%o7) \quad -15$$

(%i8) g(2); //將 x=2 代入函數 g(x)中→可得知 $f'(2)>0$

$$(\%o8) \quad \frac{15}{4}$$

$f(x)=x^2+x^{-2}$ Rewrite original function.

$f'(x)=2x-2x^{-3}$ Differentiate.

$=2x-\frac{2}{x^3}$ Rewrite with positive exponent.

$=\frac{2(x^4-1)}{x^3}$ Simplify.

$=\frac{2(x^2+1)(x-1)(x+1)}{x^3}$ Factor.

So, $f'(x)$ is zero at $x=\pm 1$. Moreover, because $x=0$ is not in the domain of f, you should use this x-value along with the critical numbers to determine the test intervals.

$x = \pm 1$ Critical numbers, $f'(\pm 1) = 0$

$x = 0$ 0 is not in the domain f.

The table summarizes the testing of the four intervals determined by these three x-values.

Interval	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = -\frac{1}{2}$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-2) < 0$	$f'(-\frac{1}{2}) > 0$	$f'(\frac{1}{2}) < 0$	$f'(2) > 0$
Conclusion	Decreasing	Increasing	Decreasing	Increasing

By applying the First Derivative Test, you can conclude that f has one relative minimum at the point (-1, 2) and another at the point (1, 2), as shown in Figure 3.22.

Example 5. The Path of a Projectile

Neglecting air resistance, the path of a projectile that is propelled at an angle θ is

$$y = \frac{g\sec^2\theta}{2{v_0}^2}x^2 + (\tan\theta)x + h, \quad 0 \le \theta \le \frac{\pi}{2}$$

where y is the height, x is the horizontal distance, g is the acceleration due to gravity, v_0 is the initial velocity, and h is the initial height. (This equation is derived in Section 12.3.) Let $g = -32$ feet per second per second, $v_0 = 24$ feet per second, and $h = 9$ feet. What value of θ will produce a maximum horizontal distance？

Solution：

To find the distance the projectile travels, let $y = 0$, and use the Quadratic Formula to solve for x.

$$\frac{g\sec^2\theta}{2{v_0}^2}x^2 + (\tan\theta)x + h = 0$$

$$\frac{-32\sec^2\theta}{2(24^2)}x^2 + (\tan\theta)x + 9 = 0$$

$$-\frac{\sec^2\theta}{36}x^2 + (\tan\theta)x + 9 = 0$$

$$x = \frac{-\tan\theta \pm \sqrt{\tan^2\theta + \sec^2\theta}}{-\sec^2\theta/18}$$

$$x = 18\cos\theta(\sin\theta + \sqrt{\sin^2\theta + 1}), \quad x \ge 0$$

At this point, you need to find the value of θ that produces a maximum value of x. Applying the First Derivative Test by hand would be very tedious. Using technology to solve the equation $dx/d\theta = 0$, however, eliminates most of the messy computations. The result is that the maximum value of x occurs when $\theta \approx 0.61548$ radian, or 35.3°.

This conclusion is reinforced by sketching the path of the projectile for different values of θ, as shown in Figure 3.23. Of the three paths shown, note that the distance traveled is greatest for $\theta = 35^\circ$.

3.4 Concavity and the Second Derivative Test

Example 1. Determining Concavity

Determine the open intervals on which the graph of $f(x) = \dfrac{6}{x^2+3}$ is concave upward or downward.

Solution：(%i1) f:6/(x^2+3); //定義一函數$\dfrac{6}{x^2+3}$，函數名稱叫做 f

(%o1) $\dfrac{6}{x^2+3}$

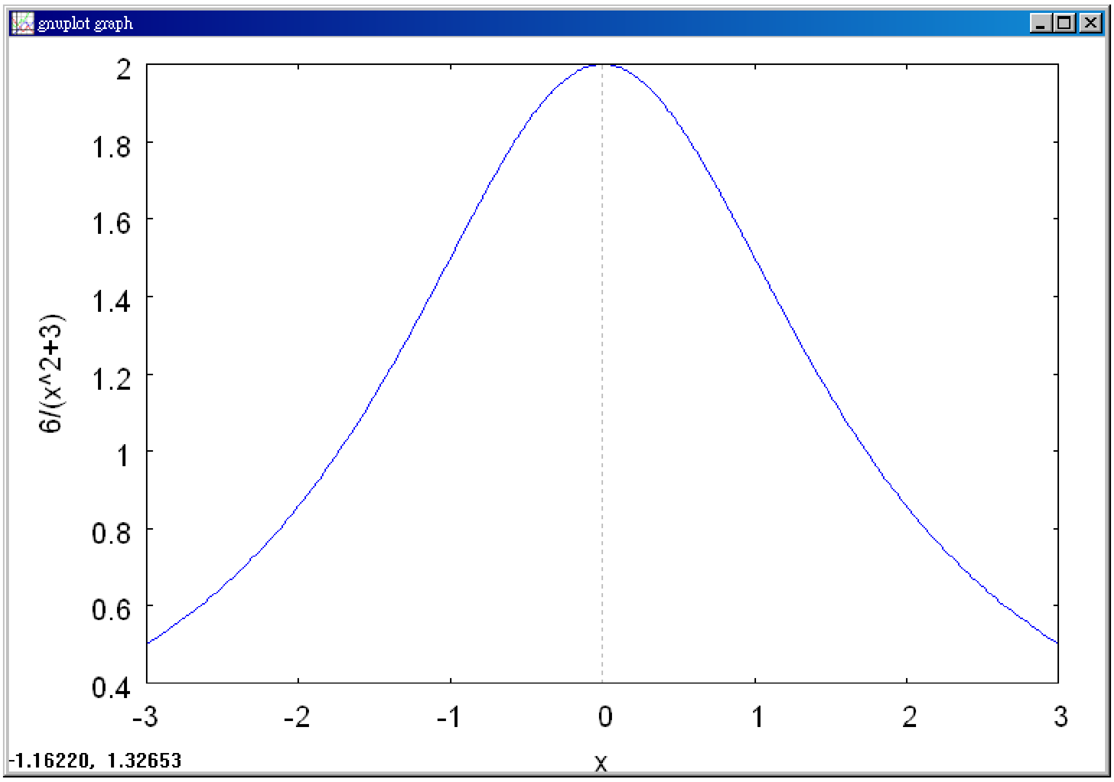
(%i2) plot2d([f],[x,-3,3]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是$\dfrac{6}{x^2+3}$函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

$$(\%o3)\quad -\frac{12\,x}{(x^2+3)^2}$$

(%i4) diff(f,x,2); 微分的指令：differ(函數，要微分的變數，次數) //對函數 f 中的 x 變數微分 2 次

$$(\%o4)\quad \frac{48\,x^2}{(x^2+3)^3}-\frac{12}{(x^2+3)^2}$$

(%i5) g(x):=48*x^2/(x^2+3)^3-12/(x^2+3)^2; //將微分 2 次後的函數名稱給定爲 g(x)

$$(\%o5)\quad g(x):=\frac{48\,x^2}{(x^2+3)^3}-\frac{12}{(x^2+3)^2}$$

(%i6) g(-2); //將 x=-2 代入函數 g(x)中→可得知 $f''(-2) > 0$

(%o6) $\frac{108}{343}$

(%i7) g(0); //將 x=0 代入函數 g(x)中→可得知 $f''(0) < 0$

(%o7) $-\frac{4}{3}$

(%i8) g(2); //將 x=2 代入函數 g(x)中→可得知 $f''(2) > 0$

(%o8) $\frac{108}{343}$

Begin by observing that f is continuous on the entire real line. Next, find the second derivative of f.

$f(x) = 6(x^2+3)^{-1}$ Rewrite original function.

$f'(x) = (-6)(x \wedge 2+3)^{-2}(2x)$ Differentiate.

$= \frac{-12}{(x^2+3)^2}$ First derivative.

$f''(x) = \frac{(x^2+3)^2(-12)-(-12x)(2)(x^2+3)(2x)}{(x^2+3)^4}$ Differentiate.

$= \frac{36(x^2-1)}{(x^2+3)^3}$ Second derivative.

Because $f''(x) = 0$ when $x = \pm 1$ and f'' is defined on the entire real line, you should test f'' in the interval (-∞, -1), (-1, 1), and (1, ∞). The results are shown in the table and in Figure 3.26.

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Example 2. Determining Concavity

Determine the open intervals on which the graph of $f(x) = \frac{x^2+1}{x^2-4}$ is concave upward or downward.

Solution：(%i1) f:(x^2+1)/(x^2-4);　//定義一函數 $\frac{x^2+1}{x^2-4}$ ，函數名稱叫做 f

(%o1) $\frac{x^2+1}{x^2-4}$

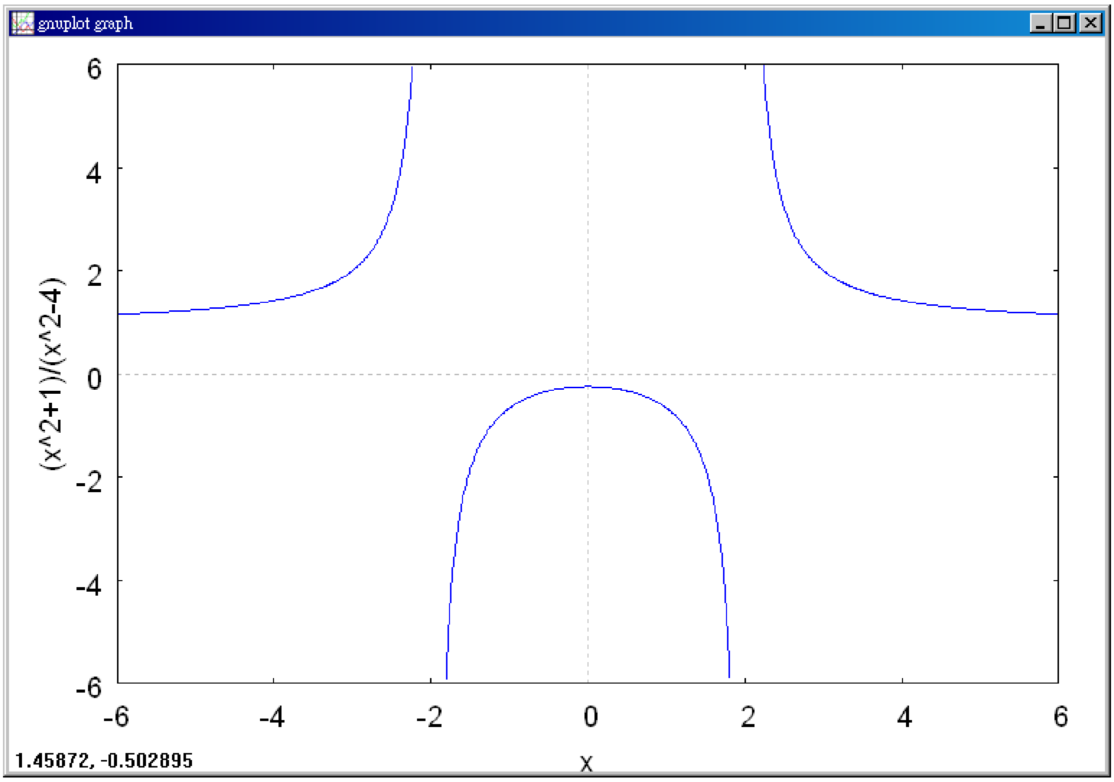
(%i2) plot2d([f],[x,-6,6],[y,-6,6]);　繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $\frac{x^2+1}{x^2-4}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-6~6，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型····等等。

```
plot2d: some values were clipped.
  (%o2)
```



(%i3) diff(f,x);　微分的指令：differ(函數，要微分的變數)　//對函數 f 中的 x 變數微分 1 次

$$(\%o3)\quad \frac{2\,x}{x^2-4}-\frac{2\,x\,(x^2+1)}{(x^2-4)^2}$$

(%i4) ratsimp(%);　有理數化簡的指令：ratsimp(數式)　//對 f 微分後的數式化簡

$$(\%o4)\quad -\frac{10\,x}{x^4-8\,x^2+16}$$

(%i5) diff(f,x,2);　微分的指令：differ(函數，要微分的變數，次數　//對函數 f 中的 x 變數微分 2 次

$$(\%o5)\quad -\frac{2\,(x^2+1)}{(x^2-4)^2}+\frac{8\,x^2\,(x^2+1)}{(x^2-4)^3}+\frac{2}{x^2-4}-\frac{8\,x^2}{(x^2-4)^2}$$

(%i6) ratsimp(%); 有理數化簡的指令：ratsimp(數式) //對 f 微分 2 次後的數式化簡

(%o6) $\dfrac{30x^2+40}{x^6-12x^4+48x^2-64}$

(%i7) g(x):=(30*x^2+40)/(x^6-12*x^4+48*x^2-64); //將微分 2 次後的函數名稱給定爲 g(x)

(%o7) $g(x):=\dfrac{30x^2+40}{x^6-12x^4+48x^2-64}$

(%i8) g(-3); //將 x=-3 代入函數 g(x)中→可得知 $f''(-3)>0$

(%o8) $\dfrac{62}{25}$

(%i9) g(0); //將 x=0 代入函數 g(x)中→可得知 $f''(0)<0$

(%o9) $-\dfrac{5}{8}$

(%i10) g(3); //將 x=3 代入函數 g(x)中→可得知 $f''(3)>0$

(%o10) $\dfrac{62}{25}$

Differentiating twice produces the following.

$$f(x)=\frac{x^2+1}{x^2-4}$$ Write original function.

$$f'(x)=\frac{(x^2-4)(2x)-(x^2+1)(2x)}{(x^2-4)^2}$$ Differentiate.

$$=\frac{-10x}{(x^2-4)^2}$$ First derivative.

$$f''(x)=\frac{(x^2-4)^2(-10)-(-10x)(2)(x^2-4)(2x)}{(x^2-4)^4}$$ Differentiate.

$$=\frac{10(3x^2+4)}{(x^2-4)^3}$$ Second derivative.

There are no points at which $f''(x)=0$, but at $x=\pm 2$ the function f is not continuous, so test for concavity in the intervals $(-\infty$, -2),(-2, 2), and (2, ∞), as shown in the table. The graph of f is shown in Figure 3.27.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Example 3. Finding Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$.

Solution：(%i1) f:x^4-4*x^3; //定義一函數 $x^4 - 4x^3$ ，函數名稱叫做 f

(%o1) $x^4 - 4\,x^3$

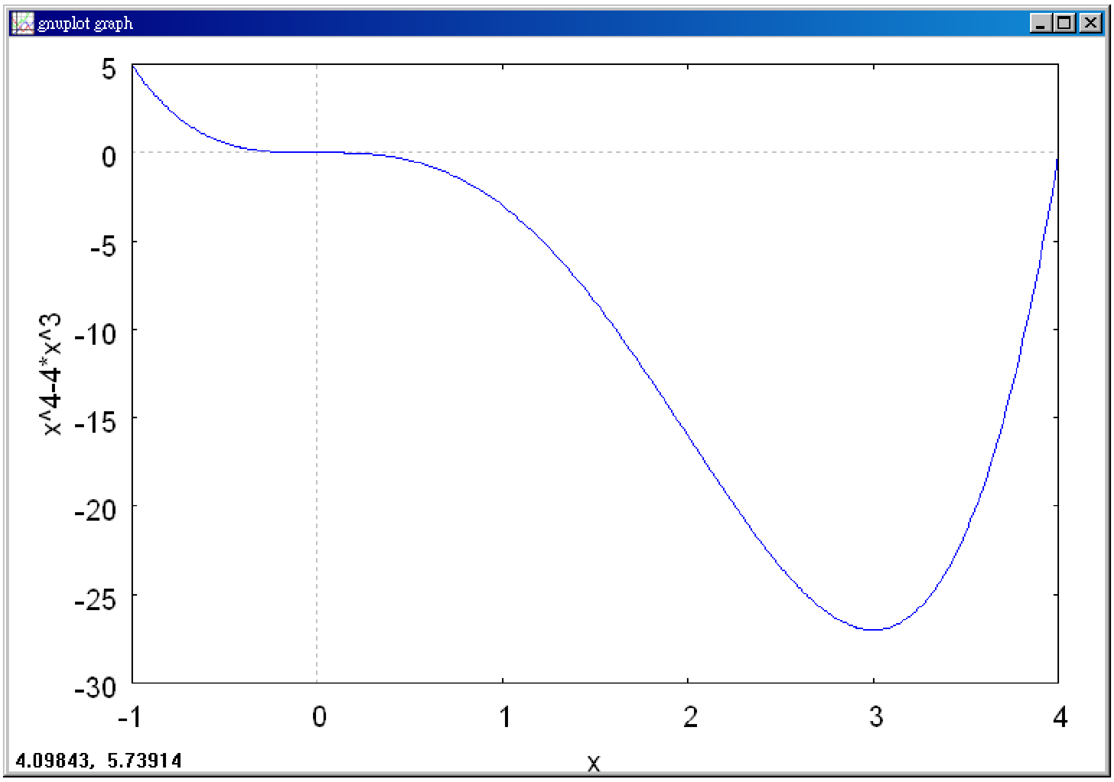
(%i2) plot2d([f],[x,-1,4]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $x^4 - 4x^3$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分 1 次

(%o3) $4\ x^3 - 12\ x^2$

(%i4) diff(f,x,2); 微分的指令：differ(函數，要微分的變數，次數) //對函數 f 中的 x 變數微分 2 次

(%o4) $12\ x^2 - 24\ x$

(%i5) g(x):=12*x^2-24*x; //將微分後的函數名稱給定爲 g(x)

(%o5) $g(x) := 12\ x^2 - 24\ x$

(%i6) g(-1); //將 x=-1 代入函數 g(x)中→可得知 $f''(-1) > 0$

(%o6) 36

(%i7) g(1); //將 x=1 代入函數 g(x)中→可得知 $f''(1) < 0$

(%o7) −12

(%i8) g(3);　//將 x=3 代入函數 g(x)中→可得知 $f''(3)>0$

(%o8) 36

Differentiating twice produces the following.

$f(x)=x^4-4x^3$ Write original function.

$f'(x)=4x^3-12x^2$ Find first derivative.

$f''(x)=12x^2-24x=12x(x-2)$ Find second derivative.

Setting $f''(x)=0$, you can determine that the possible points of inflection occur at $x=0$ and $x=2$. By testing the intervals determined by these x-values, you can conclude that they both yield points of inflection. A summary of this testing is shown in the table, and the graph of f is shown in Figure 3.29.

Interval	$-\infty<x<0$	$0<x<2$	$2<x<\infty$
Test Value	$x=-1$	$x=1$	$x=3$
Sign of $f''(x)$	$f''(-1)>0$	$f''(1)<0$	$f''(3)>0$
Conclusion	Concave upward	Concave downward	Concave upward

Example 4.　Using the Second Derivative Test

Find the relative extrema for $f(x)=-3x^5+5x^3$.

Solution：(%i1) f:-3*x^5+5*x^3;　//定義一函數 $-3x^5+5x^3$，函數名稱叫做 f

(%o1) $5x^3-3x^5$

(%i2) plot2d([f],[x,-2,2],[y,-3,3]);　繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gnuplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $-3x^5+5x^3$ 函數圖形

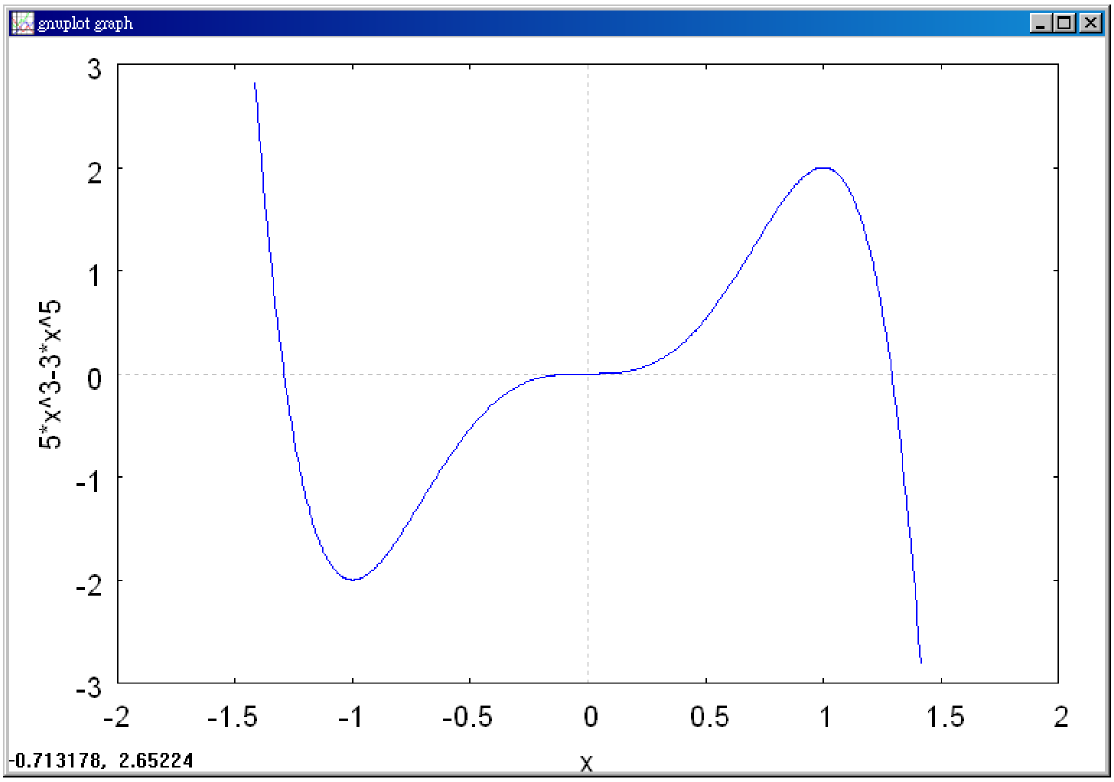
x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-3~3，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等

等。

plot2d: some values were clipped.

(%o2)



(%i3) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分 1 次

(%o3) $15\ x^2 - 15\ x^4$

(%i4) diff(f,x,2); 微分的指令：differ(函數，要微分的變數，次數) //對函數 f 中的 x 變數微分 2 次

(%o4) $30\ x - 60\ x^3$

(%i5) g(x):=30*x-60*x^3; //將微分 2 次後的函數名稱給定爲 g(x)

(%o5) $g(x) := 30\ x - 60\ x^3$

(%i6) g(-1); //將 x=-1 代入函數 g(x)中→可得知 $f''(-1) > 0$

(%o6) 30

(%i7) g(1);　//將 x=1 代入函數 g(x)中→可得知 $f''(1)\backslash < 0$

(%o7) -30

(%i8) g(0);　//將 x=-0 代入函數 g(x)中→可得知 $f''(0) = 0 =$

(%o8) 0

Begin by finding the critical numbers of f.

$f'(x) = -15x^4 + 15x^2 = 15x^2(1-x^2) = 0$　　Set $f'(x)$ equal to 0.

$x = -1, 0, 1$　　Critical numbers

Using

$f''(x) = -60x^3 + 30x = 30(-2x^3 + x)$

You can apply the Second Derivative Test as shown below.

Point	(-1, -2)	(1, 2)	(0, 0)
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(0) = 0$
Conclusion	Relative minimum	Relative maximum	Test fails

Because the Second Derivative Test fails at (0, 0), you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. So, (0, 0)is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 3.32.

3.5 Limits at Infinity

Example 1.　Finding a Limit at Infinity

Find the limit： $\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right)$.

Solution：(%i1) f:5-2/x^2;　//建立一函數 $5-\frac{2}{x^2}$ ，方程式名稱叫做 f

(%o1) $5-\frac{2}{x^2}$

(%i2) limit(f,x,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例 f 為前所定義之方程式 $5-\frac{2}{x^2}$ ，極限變數為 x，範圍為 x 趨近於 ∞

(%o2) 5

Using Theorem 3.10, you can write

$$\lim_{x\to\infty}(5-\frac{2}{x^2}) = \lim_{x\to\infty}5 - \lim_{x\to\infty}\frac{2}{x^2}$$
$$= 5 - 0$$
$$= 5.$$

Property of limits

Example 2. Finding a Limit at Infinity

Find the limit： $\lim_{x\to\infty}\frac{2x-1}{x+1}$.

Solution：(%i1) f:(2*x+1)/(x+1); //建立一函數 $\frac{2x-1}{x+1}$ ，方程式名稱叫做 f

(%o1) $\frac{2x+1}{x+1}$

(%i2) limit(f,x,inf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 $\frac{2x-1}{x+1}$ ，極限變數爲 x，範圍爲 x 趨近於∞

(%o2) 2

(%i3) plot2d([f],[x,-5,4],[y,-2,6]); 繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

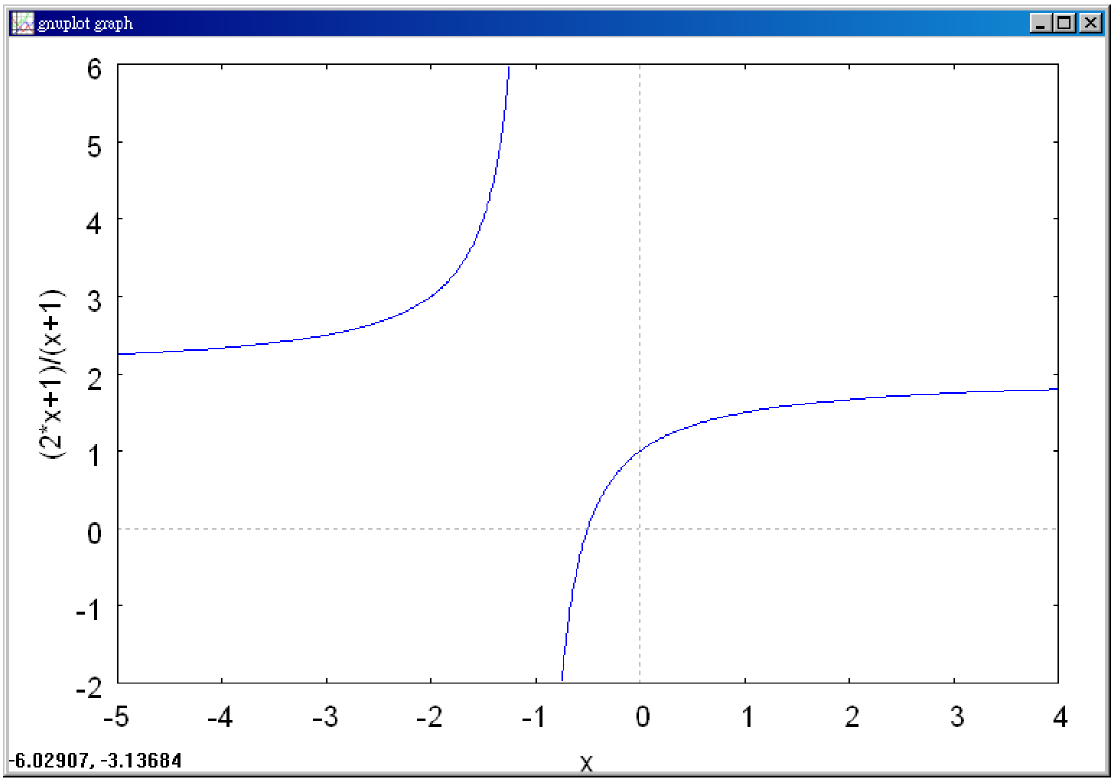
expr：是你要繪製的函數，這例是 $\frac{2x-1}{x+1}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-2~6，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

```
plot2d: some values were clipped.
```

(%o3)



Note that both the numerator and the denominator approach infinity as x approaches infinity.

$$\lim_{x\to\infty}\frac{2x-1}{x+1}\begin{cases}\lim\limits_{x\to\infty}(2x-1)\to\infty\\ \lim\limits_{x\to\infty}(x+1)\to\infty\end{cases}$$

This results in $\frac{\infty}{\infty}$, an indeterminate form. To resolve this problem, you can divide both the numerator and denominator by x. After dividing, the limit may by evaluated as shown.

$$\lim_{x\to\infty}\frac{2x-1}{x+1}=\lim_{x\to\infty}\frac{\frac{2x-1}{x}}{\frac{x+1}{x}}$$ Divide numerator and denominator by x.

$$=\lim_{x\to\infty}\frac{2-\frac{1}{x}}{1+\frac{1}{x}}$$ Simplify.

$$=\frac{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\frac{1}{x}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{1}{x}}$$ Take limits of numerator and denominator.

$$= \frac{2-0}{1+0}$$ Apply Theorem 3.10.

$$= 2$$

So, the line $y = 2$ is horizontal asymptote to the right. By taking the limit as $x \to -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left. The graph of the function is shown in Figure 3.35.

Example 3. A Comparison of Three Rational Functions

Find each limit.

a. $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$ b. $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$ c. $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

Solution：

a. (%i1) f:(2*x+5)/(3*x^2+1); //建立一函數 $\frac{2x+5}{3x^2+1}$ ，方程式名稱叫做 f

(%o1) $\frac{2\,x+5}{3\,x^2+1}$

(%i2) limit(f,x,inf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 $\frac{2x+5}{3x^2+1}$ ，極限變數爲 x，範圍爲 x 趨近於 ∞

(%o2) 0

Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty} \frac{2x+5}{3x^2+1} = \lim_{x\to\infty} \frac{(2/x)+(5/x^2)}{3+(1/x^2)} = \frac{0+0}{3+0} = \frac{0}{3} = 0$$

b. (%i1) f:(2*x^2+5)/(3*x^2+1); //建立一函數 $\frac{2x^2+5}{3x^2+1}$ ，方程式名稱叫做 f

(%o1) $\frac{2\,x^2+5}{3\,x^2+1}$

(%i2) limit(f,x,inf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 $\frac{2x^2+5}{3x^2+1}$ ，極限變數爲 x，範圍爲 x 趨近於 ∞

(%o2) $\frac{2}{3}$

Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty}\frac{2x^2+5}{3x^2+1}=\lim_{x\to\infty}\frac{2+(2/x^2)}{3+(1/x^2)}=\frac{2+0}{3+0}=\frac{2}{3}$$

c. (%i1) f:(2*x^3+5)/(3*x^2+1); //建立一函數 $\frac{2x^3+5}{3x^2+1}$，方程式名稱叫做 f

(%o1) $\frac{2\,x^3+5}{3\,x^2+1}$

(%i2) limit(f,x,inf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 $\frac{2x^3+5}{3x^2+1}$，極限變數爲 x，範圍爲 x 趨近於∞

(%o2) ∞

Divide both the numerator and the denominator by x^2.

$$\lim_{x\to\infty}\frac{2x^3+5}{3x^2+1}=\lim_{x\to\infty}\frac{2x+(5/x^2)}{3+(1/x^2)}=\frac{\infty}{3}$$

You can conclude that the limit does not exist because the numerator increases without bound while the denominator approaches 3.

Example 4. A Function with Two Horizontal Asymptotes

Find each limit.

a. $\lim_{x\to\infty}\frac{3x-2}{\sqrt{2x^2+1}}$ b. $\lim_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}}$

Solution：

a. (%i1) f:(3*x-2)/sqrt(2*x^2+1); //建立一函數 $\frac{3x-2}{\sqrt{2x^2+1}}$，方程式名稱叫做 f

(%o1) $\frac{3\,x-2}{\sqrt{2\,x^2+1}}$

(%i2) limit(f,x,inf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 $\frac{3x-2}{\sqrt{2x^2+1}}$，極限變數爲 x，範圍爲 x 趨近於∞

(%o2) $\frac{3}{\sqrt{2}}$

For $x > 0$, you can write $x = \sqrt{x^2}$. So, dividing both the numerator and the denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\frac{3x-2}{x}}{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}} = \frac{3-\frac{2}{x}}{\sqrt{\frac{2x^2+1}{x^2}}} = \frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to\infty}\frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to\infty}\frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}} = \frac{3-0}{\sqrt{2+0}} = \frac{3}{\sqrt{2}}$$

b. (%i1) f:(3*x-2)/sqrt(2*x^2+1); //建立一函數 $\frac{3x-2}{\sqrt{2x^2+1}}$ ，方程式名稱叫做 f

(%o1) $\frac{3x-2}{\sqrt{2x^2+1}}$

(%i2) limit(f,x,minf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 $\frac{3x-2}{\sqrt{2x^2+1}}$ ，極限變數爲 x，範圍爲 x 趨近於 $-\infty$

(%o2) $-\frac{3}{\sqrt{2}}$

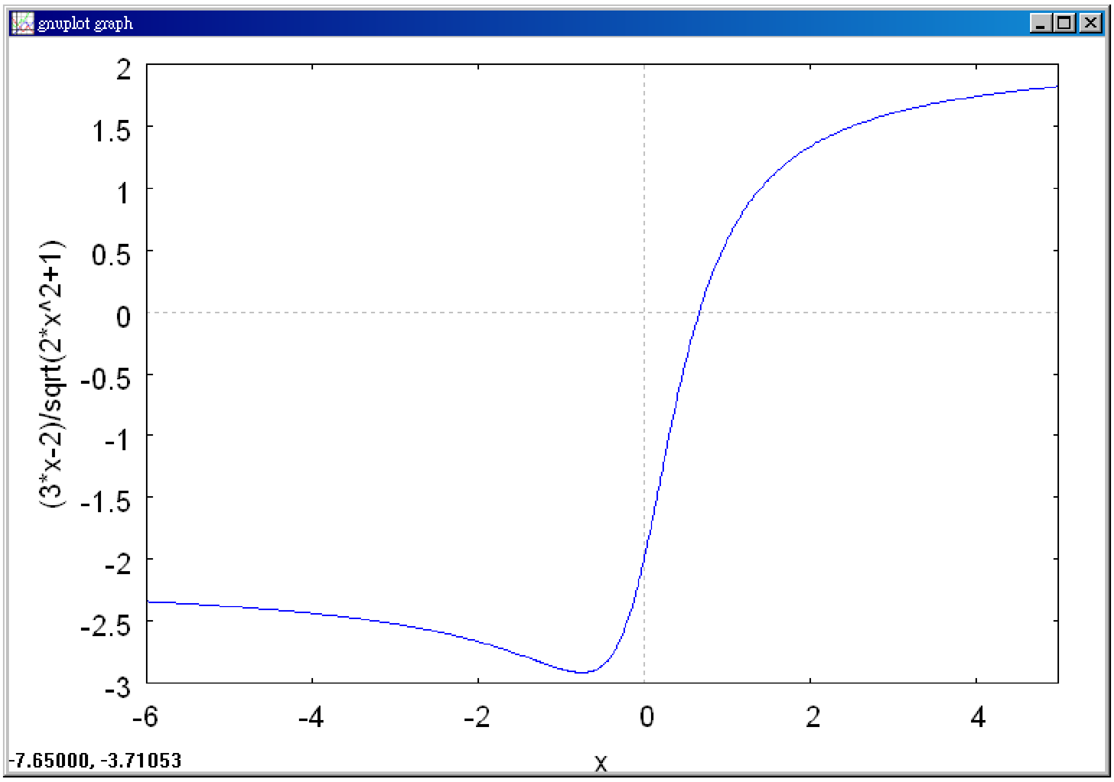
For $x < 0$, you can write $x = -\sqrt{x^2}$. So, dividing both the numerator and the denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\frac{3x-2}{x}}{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}} = \frac{3-\frac{2}{x}}{-\sqrt{\frac{2x^2+1}{x^2}}} = \frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to-\infty}\frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}} = \frac{3-0}{-\sqrt{2+0}} = -\frac{3}{\sqrt{2}}$$

The graph of $f(x) = (3x-2)/\sqrt{2x^2+1}$ is shown in Figure 3.38.



Example 5. Limits Involving Trigonometric Functions

Find each limit.

a. $\lim_{x \to \infty} \sin x$ b. $\lim_{x \to \infty} \frac{\sin x}{x}$

Solution：

a. (%i1) f:sin(x); //建立一函數 $\sin x$ ，方程式名稱叫做 f

(%o1) sin(x)

(%i2) limit(f,x,minf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 $\sin x$ ，極限變數爲 x，範圍爲 x 趨近於 ∞ ，ind 表示計算結果

indefinite but bounded

(%o2) *ind*

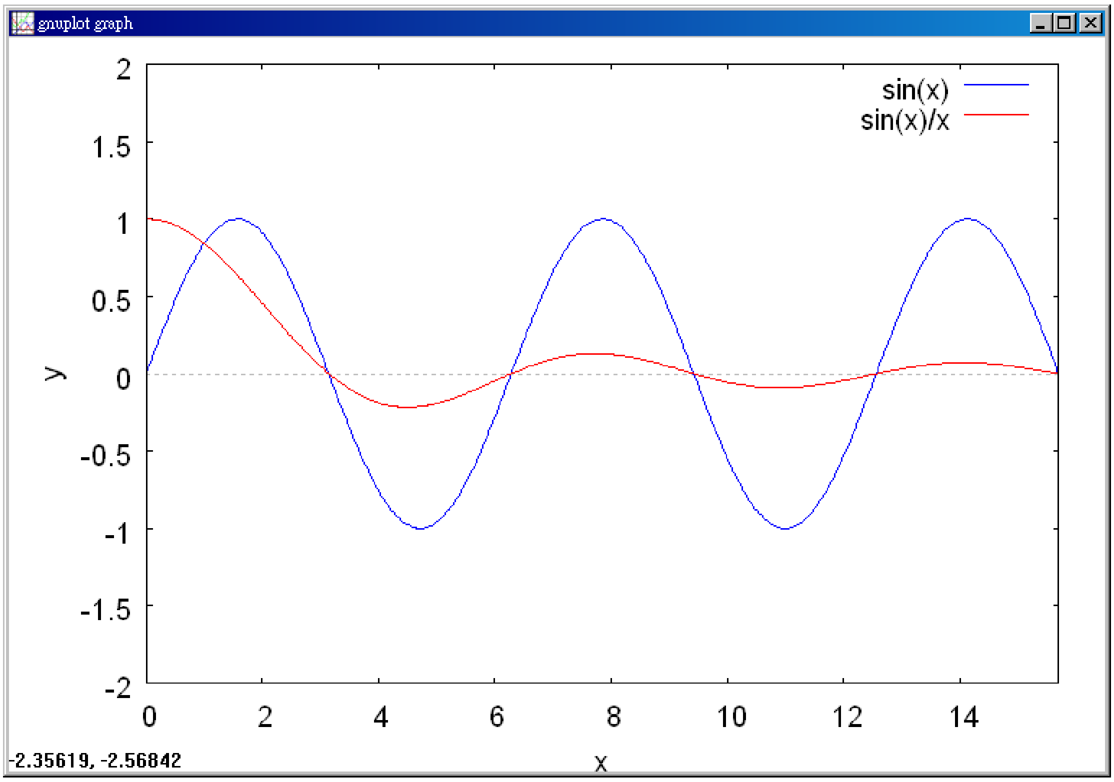
As x approaches infinity, the sine function oscillates between 1 and -1. So, this limit does not exist.

b. (%i1) f:sin(x)/x; //建立一函數$\frac{\sin x}{x}$，方程式名稱叫做 f

(%o1) $\frac{\sin(x)}{x}$

(%i2) limit(f,x,minf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式$\frac{\sin x}{x}$，極限變數爲 x，範圍爲 x 趨近於∞

(%o2) 0



Because $-1 \le \sin x \le 1$, it follows that for $x > 0$,

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

Where $\lim_{x\to\infty}(-1/x) = 0$ and $\lim_{x\to\infty}(1/x) = 0$. So, by the Squeeze Theorem, you can obtain

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

as shown in Figure 3.40.

Example 6. Oxygen Level in a Pond

Suppose that $f(t)$ measures the level of oxygen in a pond, where $f(t) = 1$ is the normal (unpolluted) level and the time t is measured in weeks. When $t = 0$, organic waste is dumped into the pond, and as the waste material oxidizes, the level of oxygen in the pond is $f(t) = \dfrac{t^2 - t + 1}{t^2 + 1}$.

What percent of the normal level of oxygen exists in the pond after 1 week？ After 2 weeks？ After 10 weeks？ What is the limit as t approaches infinity？

Solution：(%i1) f(t):=(t^2-t+1)/(t^2+1); //定義一函數 $\dfrac{t^2 - t + 1}{t^2 + 1}$ ，函數名稱叫做 f

(%o1) $f(t) := \dfrac{t^2 - t + 1}{t^2 + 1}$

(%i2) plot2d([f],[t,0,11],[y,0,1.00]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $\dfrac{t^2 - t + 1}{t^2 + 1}$ 函數圖形

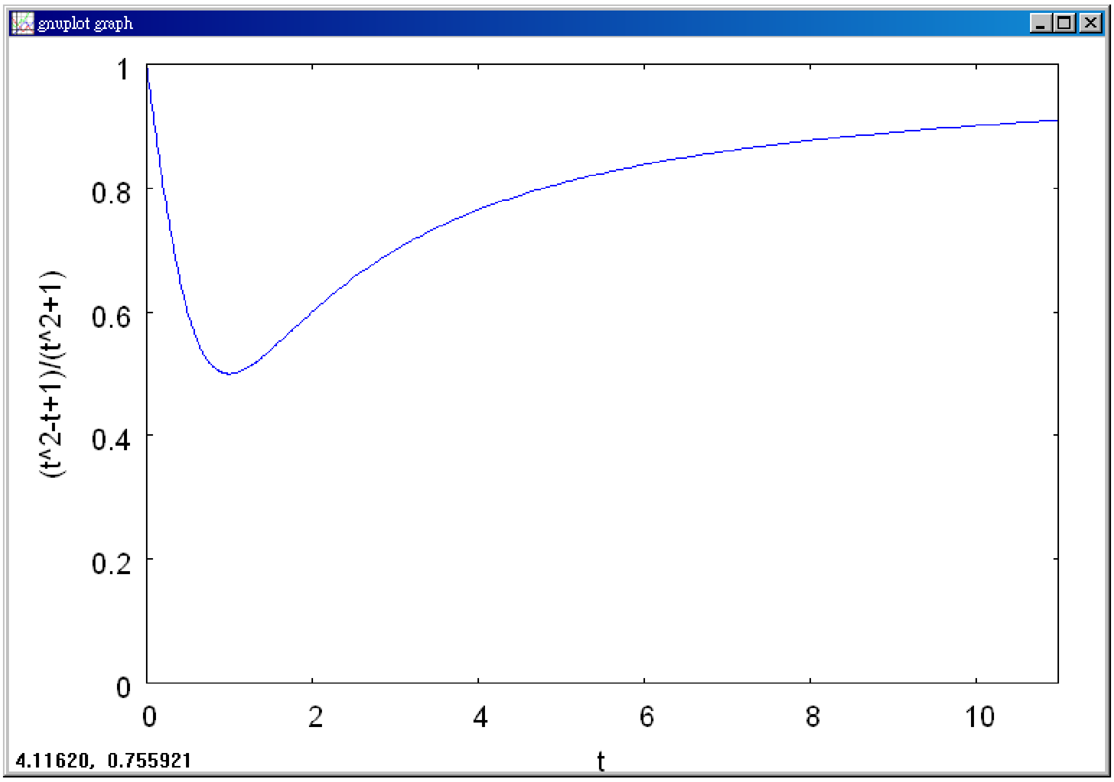
x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指

定 x 軸，這裡我們也指定了 y 軸的範圍 0~1.00，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

//The level of oxygen in a pond approaches the normal level of 1 as *t* approaches ∞

(%o2)



(%i3) f(1); //將 t=1 代入函數 f(t)中

(%o3) $\frac{1}{2}$

(%i4) f(2); //將 t=2 代入函數 f(t)中

(%o4) $\frac{3}{5}$

(%i5) f(10);　//將 t=10 代入函數 f(t)中

(%o5) $\frac{91}{101}$

(%i6) limit(f(t),t,inf);　極限指令：limit(方程式，極限變數，範圍)　//此例 f 爲前所定義之方程式 $\frac{t^2-t+1}{t^2+1}$，極限變數爲 t，範圍爲 x 趨近於∞

(%o6) 1

When $t = 1, 2$, and 10, the levels of oxygen are as shown.

$$f(1) = \frac{1^2-1+1}{1^2+1} = \frac{1}{2} = 50\%$$ 1 week

$$f(2) = \frac{2^2-2+1}{2^2+1} = \frac{3}{5} = 60\%$$ 2 week

$$f(10) = \frac{10^2-10+1}{10^2+1} = \frac{91}{101} \approx 90.1\%$$ 10 week

To find the limit as t approaches infinity, divide the numerator and the denominator by t^2 to obtain

$$\lim_{t\to\infty}\frac{t^2-t+1}{t^2+1} = \lim_{t\to\infty}\frac{1-(1/t)+(1/t^2)}{1+(1/t^2)} = \frac{1-0+0}{1+0} = 1 = 100\%$$

See Figure 3.41.

Example 7.　Finding Infinite Limits at Infinity

Find each limit.

a. $\lim_{x\to\infty} x^3$　　　　b. $\lim_{x\to-\infty} x^3$

Solution：

a. (%i1) f:x^3;　//定義一函數 x^3 ，函數名稱叫做 f

(%o1) x^3

(%i2) plot2d([f],[x,-3,3],[y,-3,3]);　繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

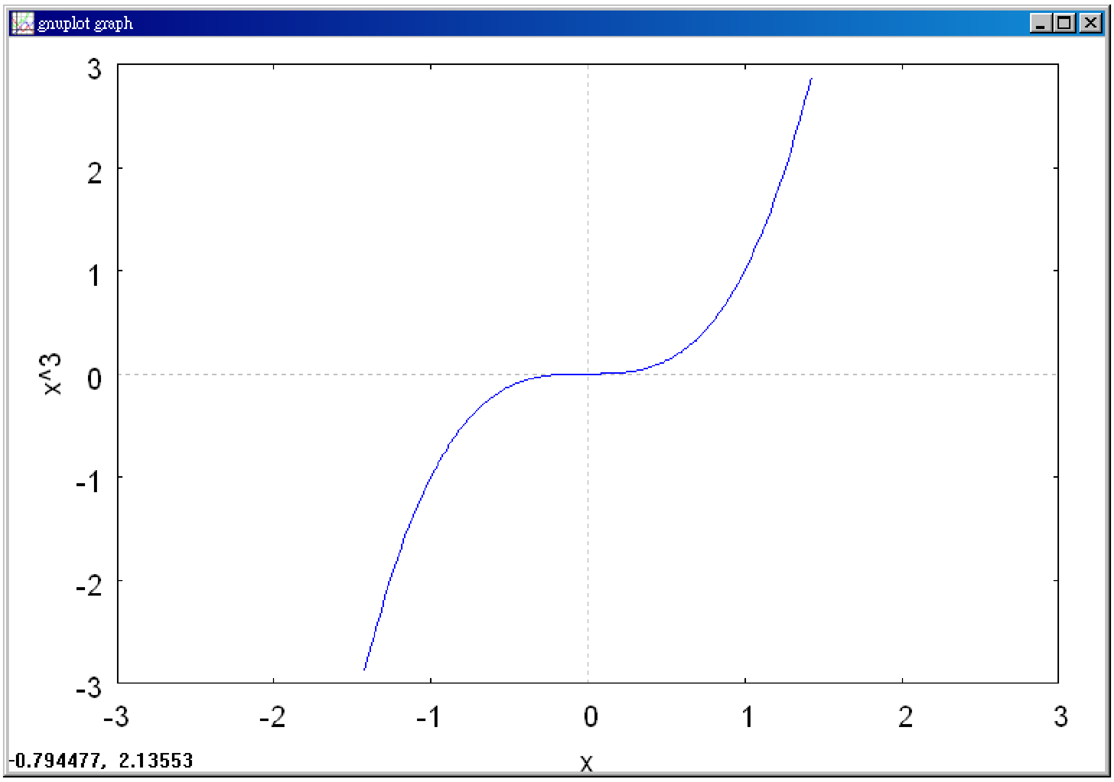
expr：是你要繪製的函數，這例是 x^3 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-3~3，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

```
plot2d: some values were clipped.
```

(%o2)



(%i3) limit(f,x,inf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 x^3 ，極限變數爲 x，範圍爲 x 趨近於 ∞

(%o3) ∞

As x increases without bound, x^3 also increases without bound. So, you can write $\lim_{x\to\infty} x^3 = \infty$.

b. (%i1) f:x^3; //建立一函數 x^3 ，方程式名稱叫做 f

(%o1) x^3

(%i2) limit(f,x,minf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 爲前所定義之方程式 x^3 ，極限變數爲 x，範圍爲 x 趨近於 $-\infty$

(%o2) $-\infty$

As x decreases without bound, x^3 also decreases without bound. So, you can write $\lim_{x\to-\infty} x^3 = -\infty$.

The graph of $f(x) = x^3$ in Figure 3.42 illustrates these two results. These results agree with the Leading Coefficient Test for polynomial functions as described in Section P.3.

Example 8. Finding Infinite Limits at Infinity

Find each limit.

a. $\lim_{x\to\infty} \frac{2x^2-4x}{x+1}$ b. $\lim_{x\to-\infty} \frac{2x^2-4x}{x+1}$

Solution：

a. (%i1) f:(2*x^2-4*x)/(x+1);　//定義一函數 $\frac{2x^2-4x}{x+1}$，函數名稱叫做 f

(%o1) $\frac{2x^2-4x}{x+1}$

(%i2) plot2d([f],[x,-15,15],[y,-20,20]);　繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

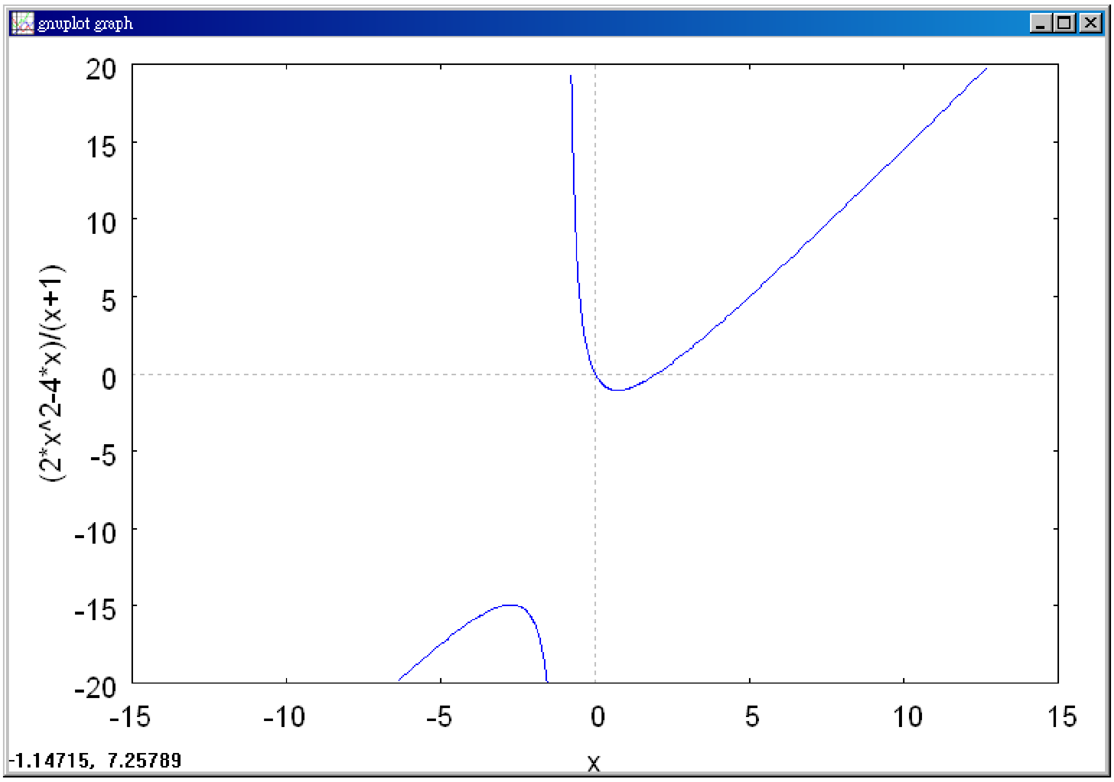
expr：是你要繪製的函數，這例是 $\frac{2x^2-4x}{x+1}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-20~20，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

```
plot2d: some values were clipped.
```

(%o2)



(%i3) limit(f,x,inf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 為前所定義之方程式 $\frac{2x^2-4x}{x+1}$，極限變數為 x，範圍為 x 趨近於 ∞

(%o3) ∞

One way to evaluate each of these limits is to use long division to rewrite the improper rational function as the sum of a polynomial and a rational function.

$$\lim_{x\to\infty}\frac{2x^2-4x}{x+1}=\lim_{x\to\infty}(2x-6+\frac{6}{x+1})=\infty$$

b. (%i1) f:(2*x^2-4*x)/(x+1); //建立一函數 $\frac{2x^2-4x}{x+1}$，方程式名稱叫做 f

(%o1) $\frac{2\,x^2-4\,x}{x+1}$

(%i2) limit(f,x,minf); 極限指令：limit(方程式，極限變數，範圍) //此例 f 為前所定義之方程式 $\frac{2x^2-4x}{x+1}$，極限變數為 x，範圍為 x 趨近於 $-\infty$

(%o2) $-\infty$

$$\lim_{x\to-\infty}\frac{2x^2-4x}{x+1}=\lim_{x\to-\infty}(2x-6+\frac{6}{x+1})=-\infty$$

The statements above can be interpreted as saying that as x approaches $\pm\infty$, the function $f(x)=(2x^2-4x)/(x+1)$ behaves like the function $g(x)=2x-6$. In Section 3.6, you will see that this is graphically described by saying that the line $y=2x-6$ is a slant asymptote of the graph of f, as shown in Figure 3.43.

3.6 A Summary of Curve Sketching

Example 1. Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x)=\dfrac{2(x^2-9)}{x^2-4}$.

Solution：(%i1) f:(2*(x^2-9))/(x^2-4); //定義一函數 $\dfrac{2(x^2-9)}{x^2-4}$ ，函數名稱叫做 f

(%o1) $\dfrac{2(x^2-9)}{x^2-4}$

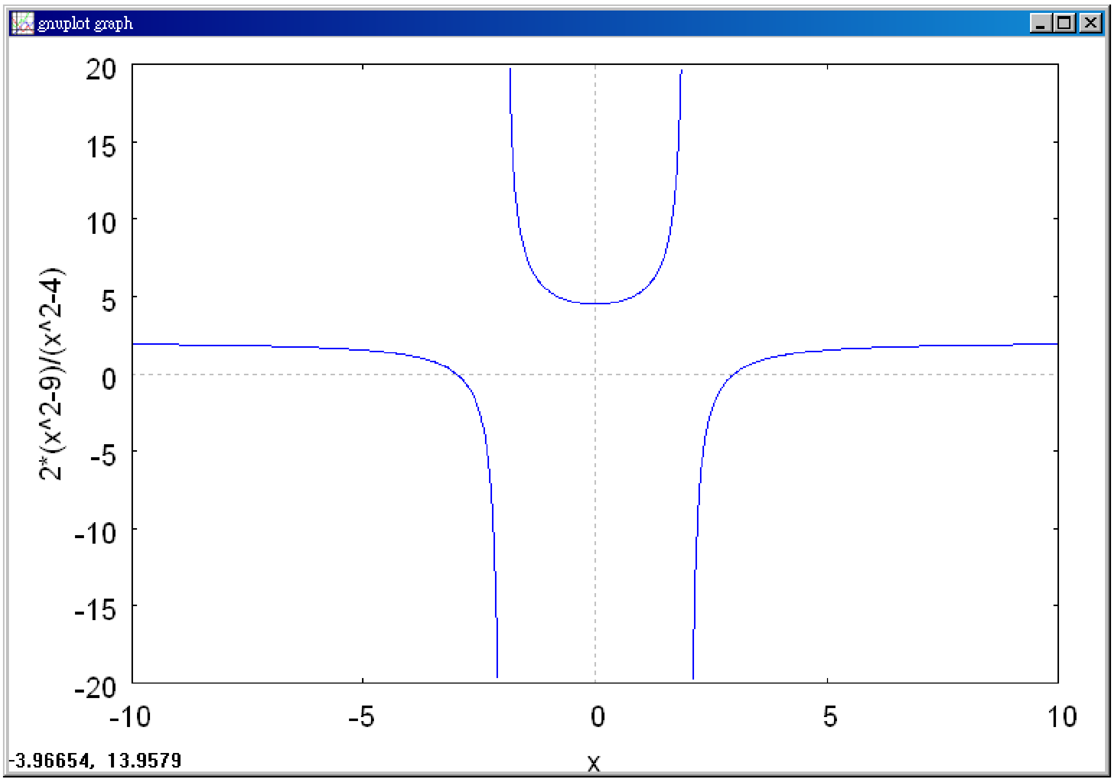
(%i2) plot2d([f],[x,-10,10],[y,-20,20]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $\dfrac{2(x^2-9)}{x^2-4}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-20~20，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

```
plot2d: some values were clipped.
  (%o2)
```



First derivative： $f'(x) = \dfrac{20x}{(x^2-4)^2}$

Second derivative： $f''(x) = \dfrac{-20(3x^2+4)}{(x^2-4)^3}$

x*-intercepts*：(-3, 0), (3, 0)

y*-intercept*：(0, $\dfrac{9}{2}$)

Vertical asymptotes： $x = -2, x = 2$

Horizontal asymptote： $y = 2$

Critical number： $x = 0$

Possible points of inflection：None

Domain：All real numbers except $x = \pm 2$

Symmetry：With respect to y-axis

Test intervals：$(-\infty,-2)$, (-2, 0), (0, 2), (2, ∞)

The table shows how the test intervals are used to determine several characteristics of the graph. The graph of f is shown in Figure 3.45.

	$f(x)$	$f'(x)$	$f''(x)$	**Characteristic of Graph**
$-\infty < x < -2$		−	−	Decreasing, concave downward
$x = -2$	Undef.	Undef.	Undef.	Vertical asymptote
$-2 < x < 0$		−	+	Decreasing, concave upward
$x = 0$	$\frac{9}{2}$	0	+	Relative minimum
$0 < x < 2$		+	+	Increasing, concave upward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < \infty$		+	−	Increasing, concave downward

Example 2.　Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

Solution： (%i1) f:(x^2-2*x+4)/(x-2);　//定義一函數 $\dfrac{x^2 - 2x + 4}{x - 2}$，函數名稱叫做 f

(%o1) $\dfrac{x^2 - 2\,x + 4}{x - 2}$

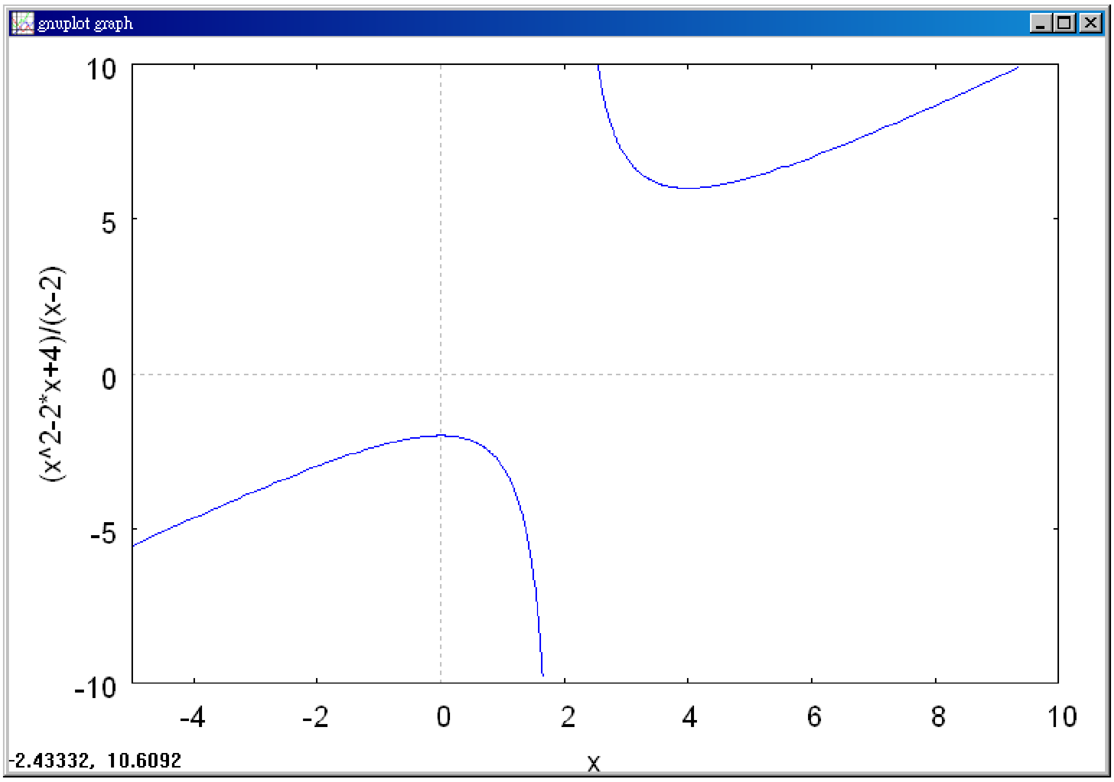
(%i2) plot2d([f],[x,-5,10],[y,-10,10]);　繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $\dfrac{x^2 - 2x + 4}{x - 2}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-10~10，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

```
plot2d: some values were clipped.
  (%o2)
```



First derivative： $f'(x)=\dfrac{x(x-4)}{(x-2)^2}$

Second derivative： $f''(x)=\dfrac{8}{(x-2)^3}$

x *-intercepts*：None

y *-intercept*：(0, -2)

Vertical asymptotes： $x=2$

Horizontal asymptote：None

End behavior： $\lim_{x\to-\infty} f(x) = -\infty, \lim_{x\to\infty} f(x) = \infty$

Critical number： $x = 0, x = 4$

Possible points of inflection：None

Domain：All real numbers except $x = 2$

Test intervals：$(-\infty, 0)$, $(0, 2)$, $(2, 4)$, $(4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.47.

	$f(x)$	$f'(x)$	$f''(x)$	**Characteristic of Graph**
$-\infty < x < 0$		+	—	Increasing, concave downward
$x = 0$	-2	0	—	Relative maximum
$0 < x < 2$		—	—	Decreasing, concave downwardward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < 4$		—	+	Decreasing, concave upward
$x = 4$	6	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave upward

Example 3. Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = \dfrac{x}{\sqrt{x^2+2}}$.

Solution：(%i1) f:x/(sqrt(x^2+2)); //定義一函數 $\dfrac{x}{\sqrt{x^2+2}}$ ，函數名稱叫做 f

(%o1) $\dfrac{x}{\sqrt{x^2+2}}$

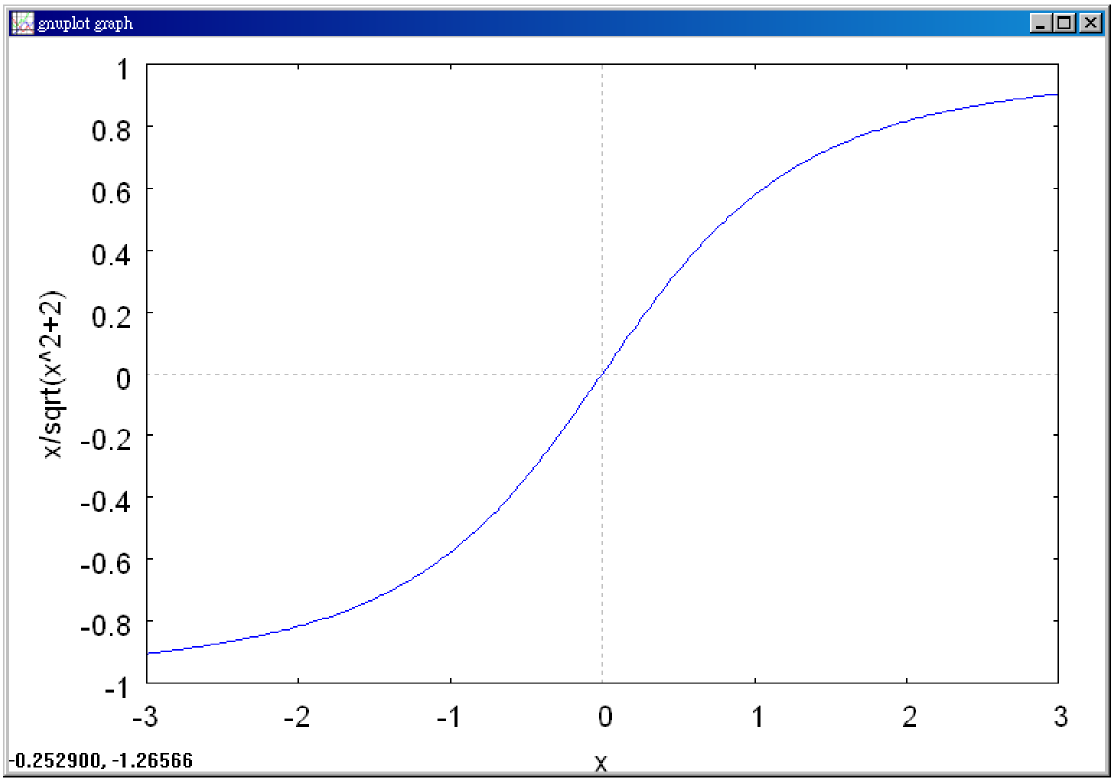
(%i2) plot2d([f],[x,-3,3]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $\dfrac{x}{\sqrt{x^2+2}}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

(%o2)



$$f'(x) = \frac{2}{(x^2+2)^{3/2}} \qquad f''(x) = -\frac{6x}{(x^2+2)^{5/2}}$$

The graph has only one intercept, (0, 0). It has no vertical asymptotes, but it has two horizontal asymptotes：$y = 1$ (to the right) and $y = -1$ (to the left). The function has no critical numbers and one possible point of inflection (at $x = 0$). The domain of the function is all real numbers, and the graph is symmetric with respect to the origin. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.49.

	$f(x)$	$f'(x)$	$f''(x)$	**Characteristic of Graph**
$-\infty < x < 0$		+	+	Increasing, concave upward
$x = 0$	0	$\frac{1}{\sqrt{2}}$	0	Point of inflection
$0 < x < \infty$		+	—	Increasing, concave downward

Example 4.　Sketching the Graph of a Radical Function

Analyze and sketch the graph of　$f(x) = 2x^{5/3} - 5x^{4/3}$.

Solution：(%i1) plot2d(2*x^(5/3)-5*x^(4/3),[x,-3,17]);　繪圖指令解說：

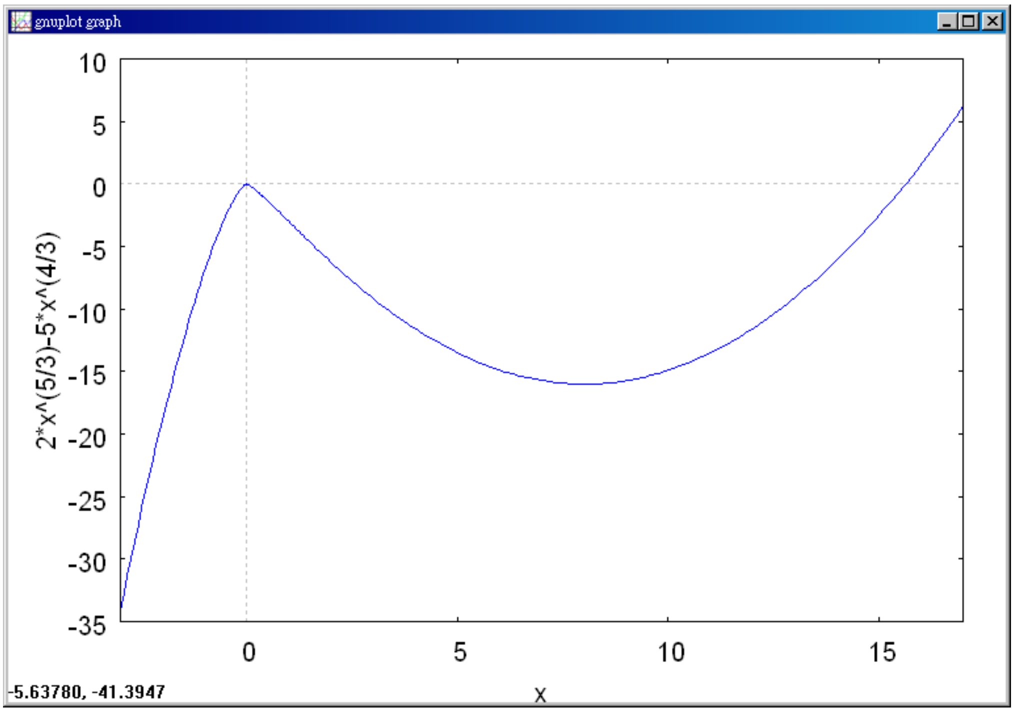
plot2d([expr，x_range，options])，　plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $2x^{5/3} - 5x^{4/3}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

(%o1)



$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \qquad f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}}$$

The function has two intercepts：(0, 0) and $(\frac{125}{8}, 0)$. There are no horizontal or vertical asymptotes. The function has two critical numbers ($x = 0$ and $x = 8$) and two possible points of inflection ($x = 0$ and $x = 1$). The domain is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.50.

	$f(x)$	$f'(x)$	$f''(x)$	**Characteristic of Graph**
$-\infty < x < 0$		+	－	Increasing, concave downward
$x = 0$	0	0	Undef.	Relative maximum
$0 < x < 1$		－	－	Decreasing, concave downward
$x = 1$	-3	－	0	Point of inflection
$1 < x < 8$		－	+	Decreasing, concave upward
$x = 8$	-16	0	+	Relative minimum
$8 < x < \infty$		+	+	Increasing, concave upward

Example 5.　Sketching the Graph of a Polynomial Function

Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

Solution：(%i1) f:x^4-12*x^3+48*x^2-64*x;　//定義一函數

$x^4 - 12x^3 + 48x^2 - 64x$ ，函數名稱叫做 f

(%o1) $x^4 - 12\,x^3 + 48\,x^2 - 64\,x$

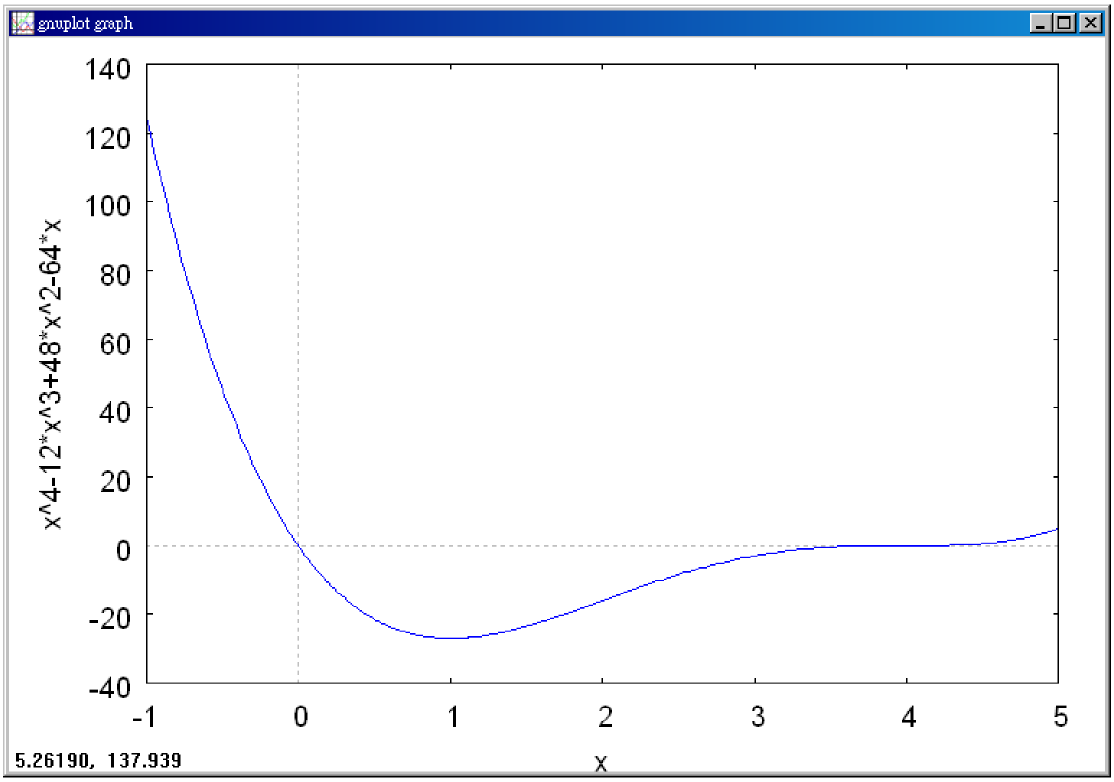
(%i2) plot2d([f],[x,-1,5]);　繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $x^4 - 12x^3 + 48x^2 - 64x$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧等等。

(%o2)



Begin by factoring to obtain $f(x) = x^4 - 12x^3 + 48x^2 - 64x = x(x-4)^3$.
Then, using the factored form of $f(x)$, you can perform the following analysis.

First derivative： $f'(x) = 4(x-1)(x-4)^2$

Second derivative： $f''(x) = 12(x-4)(x-2)$

x*-intercepts*：(0, 0), (4, 0)

y*-intercept*：(0, 0)

Vertical asymptotes：None

Horizontal asymptote：None

End behavior： $\lim_{x\to-\infty} f(x) = \infty, \lim_{x\to\infty} f(x) = \infty$

Critical number： $x = 1, x = 4$

Possible points of inflection： $x = 2, x = 4$

Domain：All real numbers

Test intervals：$(-\infty,1)$, $(1, 2)$, $(2, 4)$, $(4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.51(a). Using a computer algebra system such as Derive [see Figure 3.51(b)] can help you verify your analysis.

	$f(x)$	$f'(x)$	$f''(x)$	**Characteristic of Graph**
$-\infty < x < 1$		−	+	Decreasing, concave upward
$x = 1$	-27	0	+	Relative minimum
$1 < x < 2$		+	+	Increasing, concave upward
$x = 2$	-16	+	0	Point of inflection
$2 < x < 4$		+	−	Increasing, concave downward
$x = 4$	0	0	0	Point of inflection
$4 < x < \infty$		+	+	Increasing, concave upward

Example 6.　Sketching the Graph of a Trigonometric Function

Analyze and sketch the graph of $f(x) = \dfrac{\cos x}{1+\sin x}$.

Solution：(%i1) f:(cos(x))/(1+sin(x));　//定義一函數 $\dfrac{\cos x}{1+\sin x}$ ，函數名稱叫做 f

(%o1) $\dfrac{\cos(x)}{\sin(x)+1}$

(%i2) plot2d([f],[x,-2*%pi,2*%pi],[y,-3,3]);　繪圖指令解說：plot2d([expr，x_range，options])，　plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

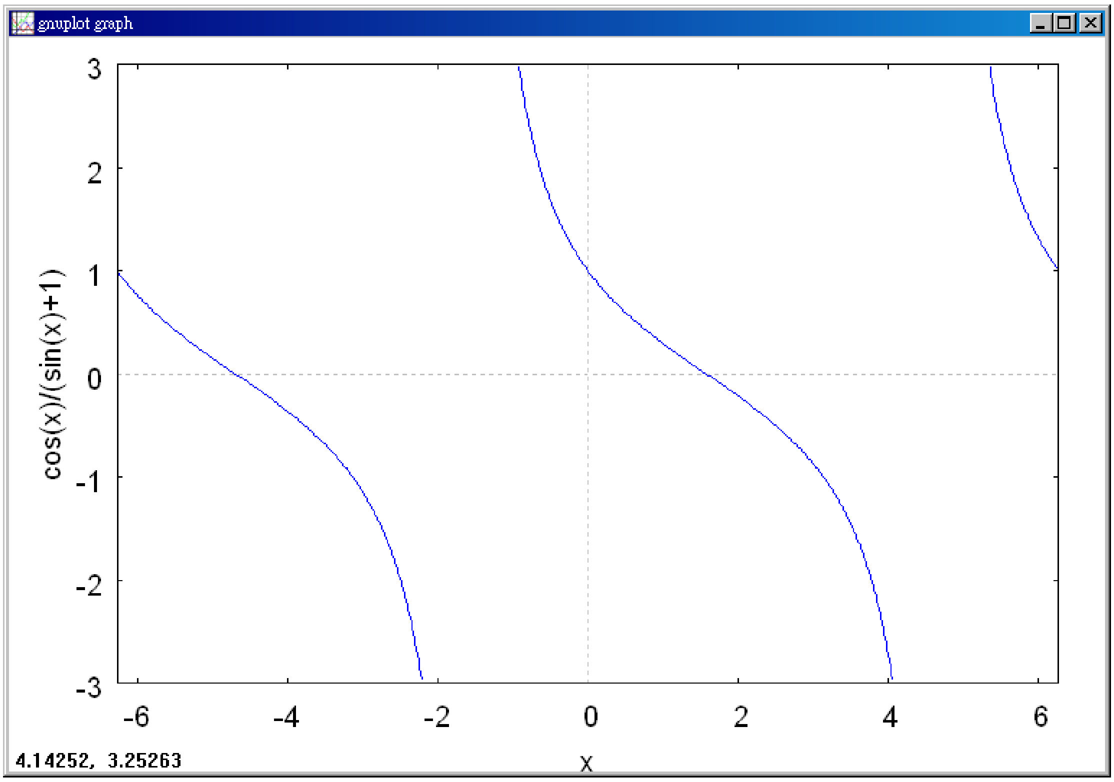
expr：是你要繪製的函數，這例是 $\dfrac{\cos x}{1+\sin x}$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，這裡我們也指定了 y 軸的範圍-3~3，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等

等。

```
plot2d: expression evaluates to non-numeric value somewhere in plott
plot2d: some values were clipped.
  (%o2)
```



Because the function has a period of 2π, you can restrict the analysis of the graph to any interval of length 2π. For convenience, choose $(-\pi/2, 3\pi/2)$.

First derivative： $f'(x) = -\dfrac{1}{1+\sin x}$

Second derivative： $f''(x) = \dfrac{\cos x}{(1+\sin x)^2}$

Period：2π

x*-intercepts*：$(\dfrac{\pi}{2}, 0)$

y*-intercept*：$(0, 1)$

Vertical asymptotes： $x=-\frac{\pi}{2}, x=\frac{3\pi}{2}$ See Note below.

Horizontal asymptote：None

Critical number：None

Possible points of inflection： $x=\frac{\pi}{2}$

Domain：All real numbers except $x=\frac{3+4n}{2}\pi$

Test intervals：$(-\frac{\pi}{2},\frac{\pi}{2}), (\frac{\pi}{2},\frac{3\pi}{2})$

The analysis of the graph of f on the interval $(-\pi/2, 3\pi/2)$ is shown in the table, and the graph is shown in Figure 3.52(a). Compare this with the graph generated by the computer algebra system Derive in Figure 3.52(b).

	$f'(x)$	$f''(x)$	$f'''(x)$	**Characteristic of Graph**
$x=-\frac{\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote
$-\frac{\pi}{2}<x<\frac{\pi}{2}$		−	+	Decreasing, concave upward
$x=\frac{\pi}{2}$	0	$-\frac{1}{2}$	0	Point of inflection
$\frac{\pi}{2}<x<\frac{3\pi}{2}$		−	−	Decreasing, concave downward
$x=\frac{3\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote

3.7 Optimization Problems

Example 1. Finding Maximum Volume

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 3.53. What dimensions will produce a box with maximum volume？

Solution：Because the box has a square base, its volume is

$V = x^2 h.$ Primary equation

This equation is called the primary equation because it gives a formula for the quantity to be optimized. The surface area of the box is

$S =$ (area of base)+(area of four sides)

$S = x^2 + 4xh = 108.$ Secondary equation

Because V is to be maximized, you want to write V as a function of just one variable. To do this, you can solve the equation $x^2 + 4xh = 108$ for h in terms of x to obtain $h = (108 - x^2)/(4x)$. Substituting into the primary equation produces

$V = x^2 h$ Function of two variables

$= x^2 (\frac{108 - x^2}{4x})$ Substitute for h

$= 27x - \frac{x^3}{4}.$ Function of one variable

Before finding which x-value will yield maximum value of V, you should determine the feasible domain. That is, what values of x make sense in this problem？ You know that $V \ge 0$. You also know that x must be nonnegative and the area of the base ($A = x^2$) is at most 108. So, the feasible domain is

$0 \le x \le \sqrt{108}.$ Feasible domain

To maximize V, find the critical numbers of the volume function.

$\frac{dV}{dx} = 27 - \frac{3x^2}{4} = 0$ Set derivative equal to 0.

$3x^2 = 108$ Simplify.

$x = \pm 6$ Critical numbers.

So, the critical numbers are $x = \pm 6$. You do not need to consider $x = -6$ because it is outside the domain. Evaluating V at the critical number $x = 6$ and at the endpoints of the domain produces $V(0) = 0, V(6) = 108$, and $V(\sqrt{108}) = 0$. So, V is maximum when $x = 6$ and the dimensions of the box are $6 \times 6 \times 3$ inches.

Example 2. Finding Minimum Distance

Which points on the graph of $y = 4 - x^2$ are closest to the point (0, 2)？

Solution：(%i11) y:4-x^2; //定義一函數 $4 - x^2$ ，函數名稱叫做 y

(%o11) $4 - x^2$

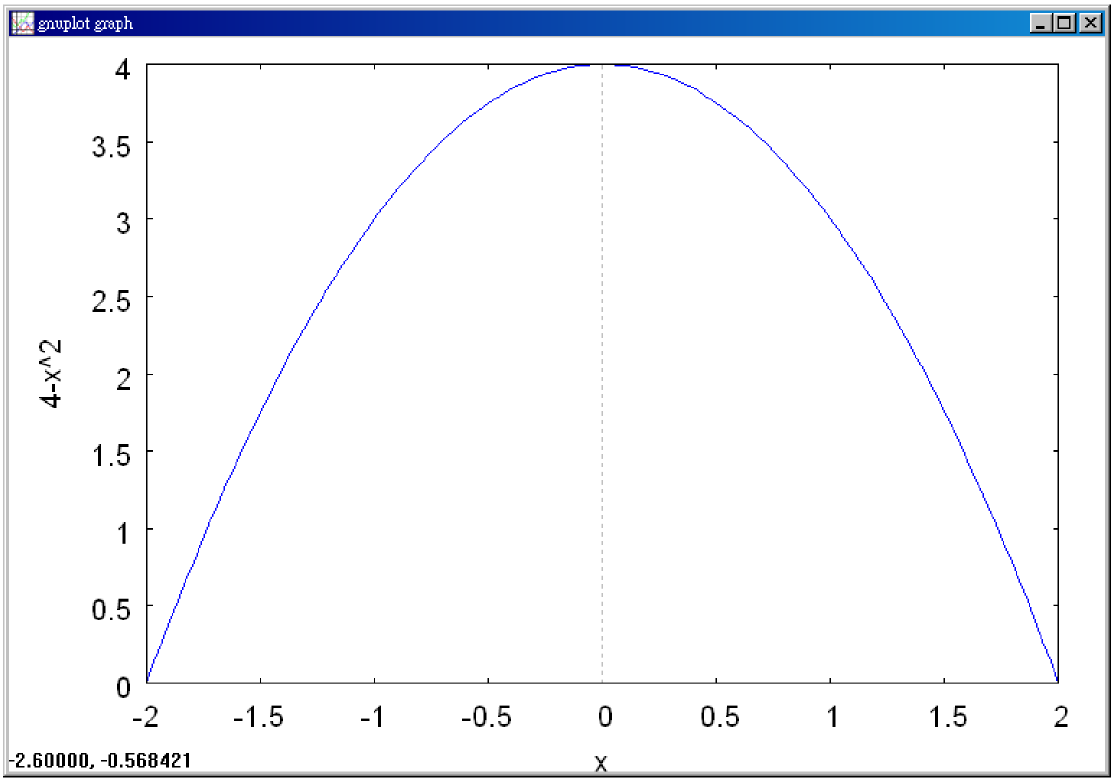
(%i12) plot2d([y],[x,-2,2]); 繪圖指令解說：plot2d([expr，x_range，options])，plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gunplot 來繪製圖形。

expr：是你要繪製的函數，這例是 $4-x^2$ 函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

(%o12)



(%i13) f:x^4-3*x^2+4; //建立一函數 x^4-3x^2+4，方程式名稱叫做 f

(%o13) $x^4-3\ x^2+4$

(%i14) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o14) $4\ x^3-6\ x$

(%i15) solve([4*x^3-6*x=0],[x]); 解方程式指令：solve([方程式],[變數]) //解 $x^4-3x^2+4=0$ 的解

(%o15) $[x=-\frac{\sqrt{3}}{\sqrt{2}}, x=\frac{\sqrt{3}}{\sqrt{2}}, x=0]$

Figure 3.55 shows that there are two points at a minimum distance from the point (0, 2). The distance between the point (0, 2) and a point (x, y) on the graph of $y=4-x^2$ is given by

$$d=\sqrt{(x-0)^2+(y-2)^2}.$$ Primary equation

Using the secondary equation $y=4-x^2$, you can rewrite the primary equation as

$$d=\sqrt{x^2+(4-x^2-2)^2}=\sqrt{x^4-3x^2+4}.$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x)=x^4-3x^2+4$. Note that the domain of f is the entire real line. So, there are no endpoints of the domain to consider. Moreover, setting $f'(x)$ equal to 0 yields

$$f'(x)=4x^3-6x=2x(2x^2-3)=0$$

$$x=0,\sqrt{\frac{3}{2}},-\sqrt{\frac{3}{2}}.$$

The First Derivative Test verifies that $x=0$ yields a relative maximum, whereas both $x=\sqrt{3/2}$ and $x=-\sqrt{3/2}$ yield a minimum distance. So, the closest points are $(\sqrt{3/2},5/2)$ and $(-\sqrt{3/2},5/2)$.

Example 3. Finding Minimum Area

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inch (see Figure 3.56). What should the dimensions of the page be so that the least amount of page is used？

Solution：(%i1) A:30+2*x+72/x; //建立一函數 $30+2x+\frac{72}{x}$ ，方程式名稱叫做 A

(%o1) $2x+\frac{72}{x}+30$

(%i2) diff(A,x); 微分的指令：differ(函數，要微分的變數) //對函數 A 中的 x 變數微分

(%o2) $2-\frac{72}{x^2}$

(%i3) solve([2-72/x^2],[x]); 解方程式指令：solve([方程式],[變數]) //解 $2-\frac{72}{x^2}=0$ 的解

(%o3) [x=-6, x=6]

Let A be the area to be minimized.

$A=(x+3)(y+2)$ Primary equation

The printed area inside the margins is given by

$24=xy$ Secondary equation

Solving this equation for y produces $y=24/x$. Substitution into the primary equation produces

$A=(x+3)(\frac{24}{x}+2)=30+2x+\frac{72}{x}$. Function of one variable

Because x must be positive, you are interested only in values of A for $x>0$. To find the critical numbers, differentiate with respect to x.

$$\frac{dA}{dx}=2-\frac{72}{x^2}=0 \Rightarrow x^2=36$$

So, the critical numbers are $x=\pm 6$. You do not have to consider $x=-6$ because it is outside the domain. The First Derivative Test confirms that A is a minimum when $x=6$. So, $y=\frac{24}{6}=4$ and the dimensions of the page should be $x+3=9$ inches by $y+2=6$ inches.

Example 4. Finding Minimum Length

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from grounded level to the top of each post. Where should the stake be placed to use the least amount of wire？

Solution：

Let W be the wire length to be minimized of z (or vice versa), you can solve for both y and z in terms of a third variable x, as shown in Figure 3.57. From the Pythagorean Theorem, you obtain

$$x^2+12^2=y^2$$

$(30-x)^2+28^2=z^2$

which implies that

$$y=\sqrt{x^2+144}$$

$$z=\sqrt{x^2-60x+1684}.$$

So, W us given by

$W=y+z$

$$=\sqrt{x^2+144}+\sqrt{x^2-60x+1684}\,,\quad 0\le x\le 30\,.$$

Differentiating W with respect to x yields

$$\frac{dW}{dx}=\frac{x}{\sqrt{x^2+144}}+\frac{x-30}{\sqrt{x^2-60x+1684}}.$$

By letting $dW/dx=0$, you obtain

$$\frac{x}{\sqrt{x^2+144}}+\frac{x-30}{\sqrt{x^2-60x+1684}}=0$$

$$x\sqrt{x^2-60x+1684}=(30-x)\sqrt{x^2+144}$$

$$x^2(x^2-60x+1684)=(30-x)^2(x^2+144)$$

$$x^4-60x^3+1684x^2=x^4-60x^3+1044x^2-8640x+129600$$

$$640x^2+8640x-129600=0$$

$$320(x-9)(2x+45)=0$$

$$x=9,-22.5$$

Because $x=-22.5$ is not in the domain and

$\mathrm{W}(0)\approx 53.04$, $\mathrm{W}(9)=50$, and $\mathrm{W}(30)\approx 60.31$

you can conclude that the wire should be staked at 9 feet from the 12-foot pole.

Example 5. An Endpoint Maximum

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area？

Solution：

The total area (see Figure 3.59) is given by

$A=$ (area of square) + (area of circle)

$A=x^2+\pi r^2$. Primary equation

So, $r=2(1-x)/\pi$, and by substituting into the primary equation you have

$$A=x^2+\pi[\frac{2(1-x)}{\pi}]^2$$

$$= x^2 + \frac{4(1-x)^2}{\pi}$$

$$= \frac{1}{\pi}[(\pi+4)x^2 - 8x + 4].$$

The feasible domain is $0 \le x \le 1$ restricted by the square's perimeter. Because

$$\frac{dA}{dx} = \frac{2(\pi+4)x - 8}{\pi}$$

the only critical number in (0, 1) is $x = 4/(\pi+4) \approx 0.56$. So, using

$A(0) \approx 1.273$, $A(0.56) \approx 0.56$, and $A(1) = 1$

you can conclude that the maximum area occurs when $x = 0$. That is, all the wire is used for the circle.

3.8 Newton's Method

Example 1. Using Newton's Method

Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

Solution：(%i1) load(newton1); 由於 Maxima 不能直接做牛頓法，因此需要讀取 newton1 此模組，此模組提供了指令去作牛頓法 //讀取 newton1 模組

```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/numeric/newton1.ma
```

(%i2) newton(x^2-2,x,1,1); 牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps //牛頓法求 $x^2 - 2$ 的解，變數爲 x，初始值爲 x=1，函數在 x 點的估計值

```
(%o2) 1.5
```

(%i3) ev(x^2-2,x=%); 指令 ev：寫一小段獨立的程式，不受其他指令干擾 //引用前面結果算出 $f(1.5)$

```
(%o3) 0.25
```

(%i4) newton(x^2-2,x,1,1/10); 牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps //牛頓法求 $x^2 - 2$ 的解，變數爲 x，初始值爲

x=1，函數在 x 點的估計值

(%o4) 1.416666666666667

(%i5) ev(x^2-2,x=%); 指令 ev：寫一小段獨立的程式，不受其他指令干擾 //引用前面結果算出 $f(1.416666666666667)$

(%o5) 0.006944444444446

(%i6) newton(x^2-2,x,1,1/100); 牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps //牛頓法求 x^2-2 的解，變數爲 x，初始值爲 x=1，函數在 x 點的估計值

(%o6) 1.416666666666667

(%i7) ev(x^2-2,x=%); 指令 ev：寫一小段獨立的程式，不受其他指令干擾 //引用前面結果算出 $f(1.416666666666667)$

(%o7) 0.006944444444446

(%i8) newton(x^2-2,x,1,1/1000); 牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps //牛頓法求 x^2-2 的解，變數爲 x，初始值爲 x=1，函數在 x 點的估計值

(%o8) 1.41421568627451

(%i9) ev(x^2-2,x=%); 指令 ev：寫一小段獨立的程式，不受其他指令干擾 //引用前面結果算出 $f(1.41421568627451)$

(%o9) $6.007304882871267\ 10^{-6}$

Because $f(x)=x^2-2$, you have $f'(x)=2x$, and the iterative process is given by the formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^{\ 2}-2}{2x_n}.$$

The calculations for three iterations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.000000	-1.000000	2.000000	-0.500000	1.500000
2	1.500000	0.250000	3.000000	0.083333	1.416667
3	1.416667	0.006945	2.833334	0.002451	1.414216
4	1.414216				

Of course, in this case you know that the zeros of the function are $\pm\sqrt{2}$. To six decimal places, $\sqrt{2} = 1.414214$. So, after only three iteration of Newton's Method, you have obtained an approximation that is within 0.000002 of an actual root. The first iteration of this process is shown in Figure 3.61.

Example 2.　Using Newton's Method

Use Newton's Method to approximate the zeros of

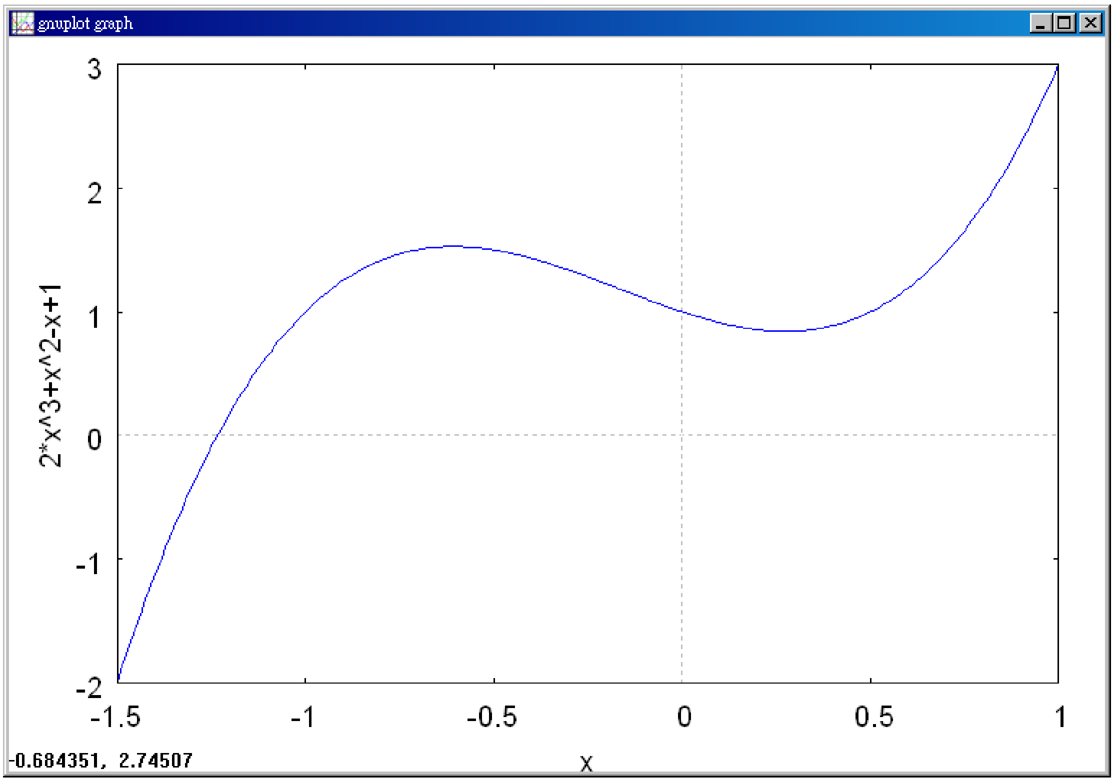
$f(x) = 2x^3 + x^2 - x + 1$.

Continue the iterations until two successive approximations differ by less than 0.0001.

Solution：(%i1) load(newton1);　由於 Maxima 不能直接做牛頓法，因此需要讀取 newton1 此模組，此模組提供了指令去作牛頓法　//讀取 newton1 模組

(%i3) plot2d(2*x^3+x^2-x+1,[x,-1.5,1]);　//畫出 $2x^3 + x^2 - x + 1$ 的圖形，x 軸範圍為-1.5~1

(%o3)



(%i2) newton(2*x^3+x^2-x+1,x,-1.2,1); 牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps //牛頓法求 $2x^3+x^2-x+1$ 的解，變數爲 x，初始值爲 x=-1.2，函數在 x 點的估計值

(%o2) -1.2

(%i3) ev(2*x^3+x^2-x+1,x=%); 指令 ev：寫一小段獨立的程式，不受其他指令干擾 //引用前面結果算出 $f(-1.2)$

(%o3) 0.184

(%i4) newton(2*x^3+x^2-x+1,x,-1.2,1/10); 牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps //牛頓法求 $2x^3+x^2-x+1$ 的解，變數爲 x，初始值爲 x=-1.2，函數在 x 點的估計值

(%o4) -1.235114503816794

(%i5) ev(2*x^3+x^2-x+1,x=%); 指令 ev：寫一小段獨立的程式，不受其他指令干擾 //引用前面結果算出 $f(-1.235114503816794)$

(%o5) -0.0077313703048496

(%i6) newton(2*x^3+x^2-x+1,x,-1.2,1/100); 牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps //牛頓法求 $2x^3+x^2-x+1$ 的解，變數爲 x，初始值爲 x=-1.2，函數在 x 點的估計值

(%o6) -1.235114503816794

(%i7) ev(2*x^3+x^2-x+1,x=%); 指令 ev：寫一小段獨立的程式，不受其他指令干擾 //引用前面結果算出 $f(-1.235114503816794)$

(%o7) -0.0077313703048496

Begin by sketching a graph of f, as shown in Figure 3.62. From the graph, you can observe that the function has only one zero, which occurs near $x=-1.2$. Next, differentiate f and form the iterative formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{2x_n^{\,3}+x_n^{\,2}-x_n+1}{6x_n^{\,2}+2x_n-1}.$$

The calculations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n-\frac{f(x_n)}{f'(x_n)}$
1	-1.20000	0.18400	5.24000	0.03511	-1.23511
2	-1.23511	-1.00771	5.68276	-0.00136	-1.23375
3	-1.23375	0.00001	5.66533	0.00000	-1.23375
4	-1.23375				

Because two successive approximations differ by less than the required 0.0001, you can estimate the zero of f to be -1.23375.

Example 3. An Example in Which Newton's Method Fails

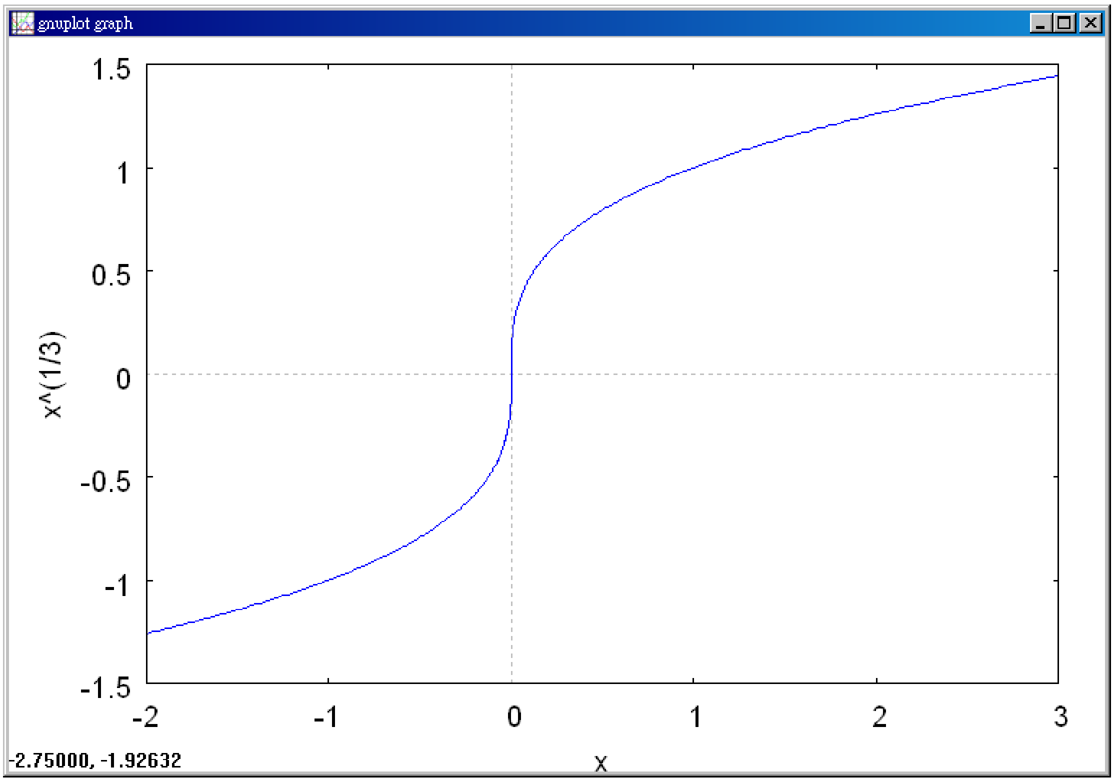
The function $f(x)=x^{1/3}$ is not differentiable at $x=0$. Show that Newton's Method fails to converge using $x_1=0.1$.

Solution：(%i1) load(newton1); 由於 Maxima 不能直接做牛頓法，因此需要讀取 newton1 此模組，此模組提供了指令去作牛頓法 //讀取 newton1 模組

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/numeric/newton1.ma

(%i2) plot2d(x^(1/3),[x,-2,3]); //畫出 $x^{1/3}$ 的圖形，x 軸為-2~3

(%o2)



(%i3) newton(x^(1/3),x,0.1,1); 牛頓法指令：newton(函數，變數，變數的初始值，eps)，函數的絕對值必須小於 eps //牛頓法求 $x^{1/3}$ 的解，變數為 x，初始值為 x=0.1，函數在 x 點的估計值

(%o3) 0.1

(%i4) ev(x^(1/3),x=%); 指令 ev：寫一小段獨立的程式，不受其他指令干擾 //引用前面結果算出 $f(0.1)$

(%o4) 0.46415888336128

Because $f'(x) = \frac{1}{3}x^{-2/3}$, the iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}$$

$$= x_n - 3x_n$$

$$= -2x_n .$$

The calculations are shown in the table. This table and Figure 3.64 indicate that x_n continuous to increase in magnitude as $n \to \infty$, and so the limit of the sequence does not exist.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.10000	0.46416	1.54720	0.30000	-0.20000
2	-0.20000	-0.58480	0.97467	-0.60000	0.40000
3	0.40000	0.73681	0.61401	1.20000	-0.80000
4	-0.80000	0.92832	0.38680	-2.40000	1.60000

3.9 Differentials

Example 1. Using a Tangent Line Approximation

Find the tangent line approximation of

$f(x) = 1 + \sin x$

at the point (0, 1). The use a table to compare the y-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.

Solution：(%i1) f:1+sin(x); //定義一函數$1+\sin x$，函數名稱叫做 f

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(%o1)  sin(x)+1
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(%i2) diff(f,x); 微分的指令：differ(函數，要微分的變數) //對函數 f 中的 x 變數微分

(%o2) cos(x)

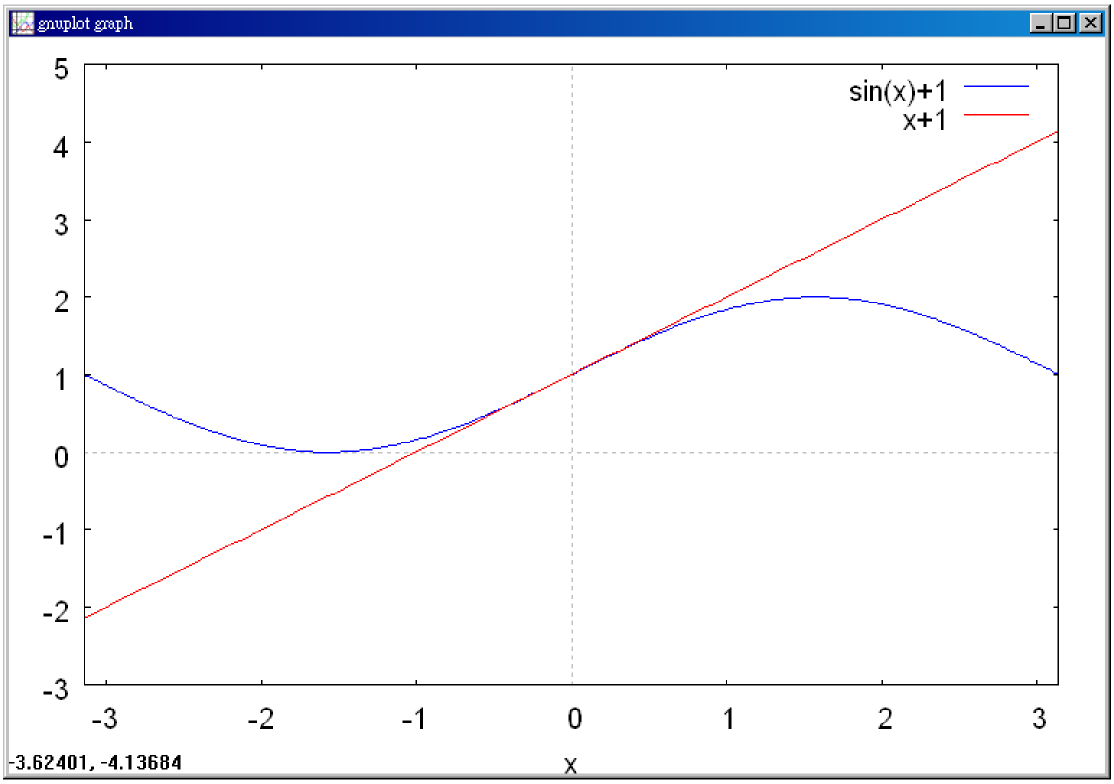
(%i3) plot2d([1+sin(x),1+x],[x,-%pi,%pi]); 繪圖指令解說：plot2d([expr，x_range，options])， plot2d 是 Maxima 的繪圖指令，maxima 執行到這時，會去呼叫 gnuplot 來繪製圖形。

expr：是你要繪製的函數，這例是$1+\sin x$函數圖形

x_range：是 x 軸的顯示範圍，當然可以指定 x 軸的顯示範圍，我們也可以指定 y 軸的顯示範圍，如果不指定 y 軸，系統也會自動設定適當的大小，不過一定要指定 x 軸，另外函數中的變數要與範圍指定的變數相同。

options：指其它的繪圖選項，如線的顏色，圖形背景色，線的大小，線型‧‧‧‧等等。

(%o3)



The derivative of f is

$f'(x) = \cos x.$ First derivative

So, the equation of the tangent line to the graph of f at the point (0, 1) is

$y - f(0) = f'(0)(x-0)$

$y - 1 = (1)(x-0)$

$y = 1 + x$. Tangent line approximation

The table compares the values of y given by this linear approximation with the values of $f(x)$ near $x = 0$. Notice that the closer x is to 0, the better the approximation is. This conclusion is reinforced by the graph shown in Figure 3.65.

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x) = 1 + \sin x$	0.521	0.9002	0.9900002	1	1.0099998	1.0998	1.479
$y = 1 + x$	0.5	0.9	0.99	1	1.01	1.1	1.5

Example 2. Comparing Δy and dy

Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

Solution： Because $y = f(x) = x^2$, you have $f'(x) = 2x$, and the differential dy is given by

$dy = f'(x)dx = f'(1)(0.01) = 2(0.01) = 0.02$. Differential of y

Now, using $\Delta y = f(x + \Delta x) - f(x) = f(1.01) - f(1) = (1.01)^2 - 1^2 = 0.0201$.

Figure 3.67 shows the geometric comparison of dy and Δy. Try comparing other values of dy and Δy. You will see that the values becomes closer to each other as dx (or Δx) approaches 0.

Example 3. Estimation of Error

The radius of a ball bearing is measured to be 0.7 inch, as shown in Figure 3.68. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

Solution：The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. So, you can write

$r = 0.7$ Measured radius

and

$-0.01 \le \Delta r \le 0.01$ Possible error

To approximate the propagated error in the volume, differentiate V to obtain $dV/dr = 4\pi r^2$ and write

$\Delta V \approx dV$ Approximate ΔV by dV.

$= 4\pi r^2 dr$

$= 4\pi (0.7)^2 (\pm 0.01)$ Substitute for r and dr.

$\approx \pm 0.06158$ cubic inch.

So, the volume has a propagated error of about 0.06 cubic inch.

Example 4. Finding Differentials

Function	Derivative	Differential
a. $y = x^2$	$\frac{dy}{dx} = 2x$	$dy = 2xdx$
b. $y = 2\sin x$	$\frac{dy}{dx} = 2\cos x$	$dy = 2\cos xdx$
c. $y = x\cos x$	$\frac{dy}{dx} = -x\sin x + \cos x$	$dy = (-x\sin x + \cos x)$
d. $y = \frac{1}{x}$	$\frac{dy}{dx} = -\frac{1}{x^2}$	$dy = -\frac{dx}{x^2}$

Example 5. Finding the Differential of a Composite Function

$y = f(x) = \sin 3x$ Original function

$f'(x) = 3\cos 3x$ Apply Chain Rule.

$dy = f'(x)dx = 3\cos 3xdx$ .Differential form

Example 6. Finding the Differential of a Composite Function

$y = f(x) = (x^2 + 1)^{1/2}$ Original function

$f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}}$ Apply Chain Rule.

$dy = f'(x)dx = \frac{x}{\sqrt{x^2 + 1}}dx$ Differential form

Example 7. Approximating Function Values

Use differentials to approximate $\sqrt{16.5}$.

Solution：Using $f(x) = \sqrt{x}$, you can write

$$f(x + \Delta x) \approx f(x) + f'(x) = \sqrt{x} + \frac{1}{2\sqrt{x}}dx.$$

Now, choosing $x = 16$ and $dx = 0.5$, you obtain the following approximation.