http://www.npue.edu.tw/academic/math/index.htm

Ch 2.1 The Derivative and the Tangent Line Problem

Example 1 The slope of the Graph of a Linear Function

Find the slope of the graph of

$$f(x) = 2x - 3$$

at the point (2, 1).

Sol:

To find the slope of the graph of f when $c = 2$, you can apply the definition of the slope of a tangent line, as shown.

$$\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[2(2+\Delta x) - 3] - [2(2) - 3]}{\Delta x}$$

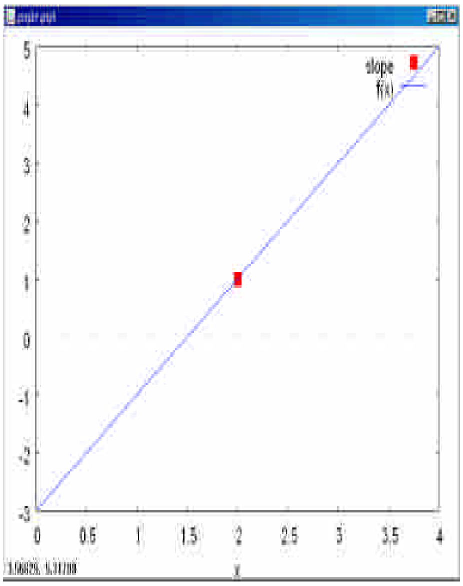
$$= \lim_{\Delta x \to 0} \frac{4 + 2\Delta x - 3 - 4 + 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2$$

$$= 2$$

The slope of f at $(c, f(c)) = (2,1)$ is m =2, as shown in Figure 2.5.



(%i53) f(x):=2*x-3;

(%o53) f(x):=2 x−3

(%i56) diff(f(x),x);

(%o56) 2

(%i64) xy:[[2,1]]$

(%i65) plot2d([2*x-3],[x,0,4]);

(%o65)

(%i66) plot2d([discrete, xy], [style,points])$

(%i70) plot2d([[discrete, xy],2*x-3],[x,0,4],[style, [points,5,2,6], [lines,1,1]], [legend,"slope","f(x)"]);

(%o70)

Example 2 Tangent Lines to the Graph of a Nonlinear Function

Find the slope of the tangent lines to the graph of

$$f(x) = x^2 + 1$$

at the points (0, 1) and (-1, 2), as shown in Figure 2.6.

Sol:

Let $(c, f(c))$ represent an arbitrary point on the graph of f. Then the slope of the tangent line at $(c, f(c))$ is given by

$$\lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(c+\Delta x)^2 + 1] - (c^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c^2 + 2c(\Delta x) + (\Delta x)^2 + 1 - c^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2c(\Delta x) + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2c + \Delta x)$$

$$= 2c$$

So, the slope at any point $(c, f(c))$ on the graph of f is $m = 2c$. At the point (0, 1), the slope is $m = 2(0) = 0$, and at (-1, 2), the slope is $m = 2(-1) = -2$.

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(%i94) f(x):=x^2+1;

(%o94) $f(x):=x^2+1$

(%i95) diff(f(x),x);

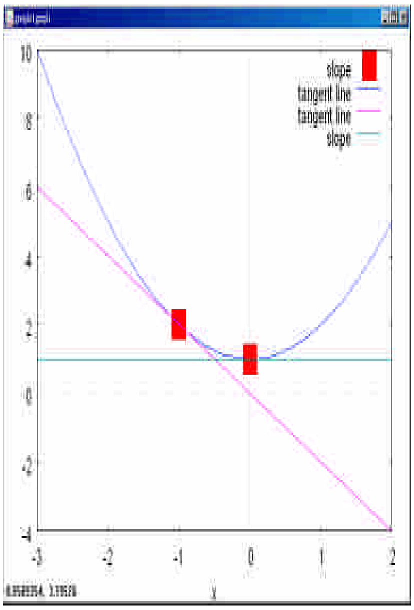
(%o95) 2 x

(%i96) f(x):=2*x;

(%o96) f(x):=2 x

(%i99) f(-1);

(%o99) -2



(%i75) g(x):=-(2*x);

(%o75) g(x):=-2 x

(%i76) plot2d(g(x),[x,-3,1]);

(%o76)

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```
(%i77) f(x):=x^2+1;
(%o77) f(x):=x^2+1

(%i78) plot2d(f(x),[x,-2,2]);
(%o78)

(%i79) h(x):=1;
(%o79) h(x):=1

(%i80) plot2d(h(x),[x,-2,2]);
(%o80)

(%i81) plot2d([f,g,h],[x,-3,2]);
(%o81)

(%i82) xy:[[0,1],[-1,2]]$

(%i83) plot2d([discrete, xy], [style,points])$

(%i93) plot2d([[discrete, xy],x^2+1,-(2*x),1],[x,-3,2],[style, [points,5,2,6],
[lines,1,1],[line,1,1],[line,1,1]],
[legend,"slope","tangent line","tangent line"]);
```

Example3 Finding the Derivative by the Limit Process

Find the derivative of $f(x)=x^3+2x$

Sol:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$ Definition of derivative

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3+2(x+\Delta x)-(x^3+2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2]$$

$$= 3x^2 + 2$$

```
(%i100) f(x):=x^3+2*x;
```

(%o100) $f(x) := x^3 + 2\,x$

```
(%i101) diff(f(x),x);
```

(%o101) $3\,x^2 + 2$

Example 4 Using the Derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x) = \sqrt{x}$ .Then find the slope of the graph of f at the points (1,1) and (4, 2). Discuss the behavior of f at (0, 0).

Sol:

Use the procedure for rationalizing numerators, as discussed in Section 1.3.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$ Definition of derivative

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \right) \left(\frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{(x + \Delta x) - x}{\Delta x \sqrt{x + \Delta x} + \sqrt{x}} \right)$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta x}{\Delta x \sqrt{x + \Delta x} + \sqrt{x}} \right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} \qquad x > 0$$

At the point (1, 1), the slope is $f'(1) = \frac{1}{2}$. At the point (4 ,2), the slope is $f'(4) = \frac{1}{4}$. See Figure 2.8. At the point (0, 0), the slope is undefined. Moreover, the graph of f has a vertical tangent line at (0, 0).

Example 5 Finding the Derivative of a Function

Find the derivative with respect to t for the function $y = 2/t$.

Sol:

Considering $y = f(t)$, you obtain

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{2}{t+\Delta t} - \frac{2}{t}}{\Delta t} \qquad f(t+\Delta t) = 2/(t+\Delta t) \text{ and } f(t) = 2/t$$

$$= \lim_{\Delta t \to 0} \frac{\frac{2t - 2(t+\Delta t)}{t(t+\Delta t)}}{\Delta t} \qquad \text{Combine fractions in numerator.}$$

$$= \lim_{\Delta t \to 0} \frac{-2\Delta t}{\Delta t(t)(t+\Delta t)} \qquad \text{Divide out common factor of } \Delta t.$$

$$= \lim_{\Delta t \to 0} \frac{-2}{t(t+\Delta t)} \qquad \text{Simplify.}$$

$$= -\frac{2}{t^2} \qquad \text{Evaluate limit as } \Delta t \to 0.$$

```
(%i102) f(t):=2/t;
```

$$\text{(\%o102)} \quad f(t) := \frac{2}{t}$$

```
(%i103) diff(f(t),t);
```

$$\text{(\%o103)} \quad -\frac{2}{t^2}$$

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Example 6 A Graph with a Sharp Turn

The function

$$f(x) = |x-2|$$

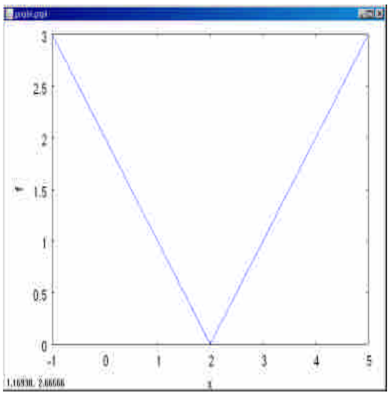
Shown in Figure 2.12 is continuous at $x = 2$. But, the one-sided limits

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{|x-2|-0}{x-2} = -1$$ Derivative from the left

and

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{|x-2|-0}{x-2} = 1$$ Derivative from the right

are not equal. So, f is not differentiable at $x = 2$ and the graph of f does not have a tangent line at the point (2, 0).



(%i123) f(x):=abs(x-2);

plot2d([f],[x,-1,5]);

(%o123) $f(x) := |x-2|$

Example 7 A Graph with a Vertical Tangent Line

The function

$$f(x) = x^{1/3}$$

is continuous at $x = 0$, as shown in Figure 2.13. But, because the limit

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^{1/3}-0}{x}$$

$$= \lim_{x\to 0} \frac{1}{x^{2/3}}$$

$= \infty$

is infinite, you can conclude that the tangent line is vertical at $x = 0$. So, f is not differentiable at $x = 0$.

Ch 2.2 Basic Differentiation and the Tangent Line Problem

Example 1 Using the Constant Rule

Function	Derivative
a. $y = 7$	$\dfrac{dy}{dx} = 0$
b. $f(x) = 0$	$f'(x) = 0$
c. $s(t) = -3$	$s'(x) = 0$
d. $y = k\pi^2$, k is constant	$y' = 0$

(%i2) y:7;

(%o2) 7

(%i3) diff(y);

(%o3) 0

(%i4) f(x):=0;

(%o4) f(x):=0

(%i6) diff(f(x));

(%o6) 0

(%i7) s(t):=-3;

(%o7) s(t):=-3

(%i8) diff(s(t));

(%o8) 0

(%i16) declare(k,constant);

(%o16) $done$

(%i17) y:((%pi)^2)*k;

(%o17) $\pi^2 k$

(%i18) diff(y);

(%o18) 0

Example 2 Using the Power Rule

Function	Derivative
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
c. $y = \frac{1}{x^2}$	$\frac{dy}{dx} = \frac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\frac{2}{x^3}$

(%i19) f(x):=x^3;

(%o19) $f(x):=x^3$

(%i23) diff(f(x),x);

(%o23) $3\,x^2$

(%i24) g(x):=x^(1/3);

(%o24) $g(x):=x^{1/3}$

(%i25) diff(g(x),x);

(%o25) $\frac{1}{3\,x^{2/3}}$

(%i26) y:1/(x^2);

(%o26) $\frac{1}{x^2}$

(%i27) diff(y,x);

(%o27) $-\frac{2}{x^3}$

Example 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^4$ when

a. $x = -1$ b. $x = 0$ c. $x = 1$

Sol:

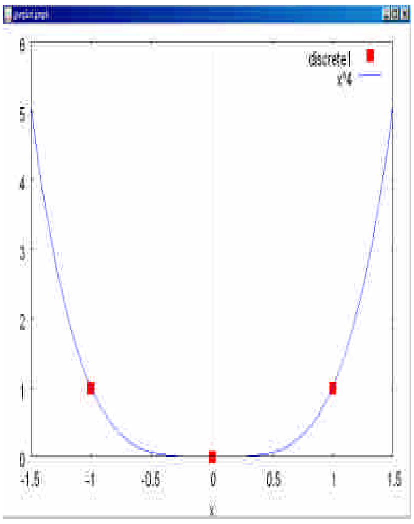
The slope of a graph at a point is the value of the derivative at that point. The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$ Slope is negative.

b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$ Slope is zero.

c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$ Slope is positive.

See Figure 2.16.



(%i28) f(x):=x^4;

(%o28) $f(x):=x^4$

(%i29) diff(f(x),x);

(%o29) $4\ x^3$

(%i30) f(x):=4*x^3;

(%o30) $f(x):=4\ x^3$

(%i31) f(-1);

(%o31) -4

(%i32) f(0);

(%o32) 0

(%i33) f(1);

(%o33) 4

(%i34) f(x):=x^4;

(%o34) $f(x):=x^4$

(%i36) plot2d(f,[x,-2,2]);

(%o36)

(%i45) xy:[[1,1],[-1,1],[0,0]]$

(%i46) plot2d([discrete, xy], [style,points])$

(%i47) plot2d([[discrete,xy], x^4], [x,-1.5,1.5], [style, [points,5,2,6], [lines,1,1]])$

Example 4 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Sol:

To find the point on the graph of f, evaluate the original function at $x = -2$

$(-2, f(-2)) = (-2, 4)$ Point on graph

To find the slope of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$

$m = f'(-2) = -4$ Slope of graph at (-2, 4)

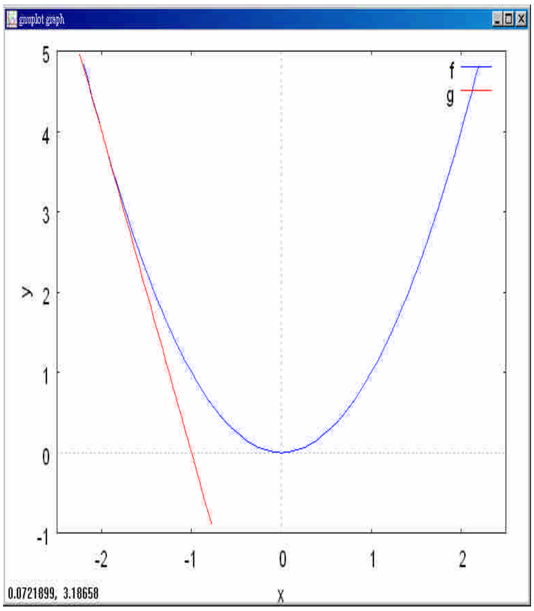
Now, using the point-slope form of the equation of a line, you can write

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 4 = -4[x - (-2)]$ Substitute for y_1, m, and x_1.

$y = -4x - 4$ Simplify.

See Figure 2.17.



(%i1) f(x):=x^2;

(%o1) $f(x):=x^2$

(%i3) g(x):=-4*x-4;

(%o3) $g(x):=(-4)x-4$

(%i6) plot2d([f,g],[x,-2.5,2.5],[y,-1,5]);

```
plot2d: some values were clipped.
plot2d: some values were clipped.
```

(%o6)

(%i7) f(-2);

(%o7) 4

(%i8) diff(f(x),x);

(%o8) 2 x

(%i9) f(m):=m*(x-x1)+y1;

(%o9) f(m):=m(x−x1)+y1

(%i10) f(-4);

(%o10) y1−4(x−x1)

(%i11) f(x1,y1):=y1+(-4)*(x-x1);

(%o11) f(x1,y1):=y1+(−4)(x−x1)

(%i12) f(-2,4);

(%o12) 4−4(x+2)

Example 5 Using the Constant Multiple Rule

Function	Derivative
a. $y=\frac{2}{x}$	$\frac{dy}{dx}=\frac{d}{dx}[2x^{-1}]=2\frac{d}{dx}[x^{-1}]=2(-1)x^{-2}=-\frac{2}{x^2}$
b. $f(t)=\frac{4t^2}{5}$	$f'(t)=\frac{d}{dt}[\frac{4}{5}t^2]=\frac{4}{5}\frac{d}{dt}[t^2]=\frac{4}{5}(2t)=\frac{8}{5}t$
c. $y=2\sqrt{x}$	$\frac{dy}{dx}=\frac{d}{dx}[2x^{1/2}]=2(\frac{1}{2}x^{-1/2})=x^{-1/2}=-\frac{1}{\sqrt{x}}$

d. $y = \frac{1}{2\sqrt[3]{x^2}}$ $\frac{dy}{dx} = \frac{d}{dx}[\frac{1}{2}x^{-2/3}] = \frac{1}{2}(-\frac{2}{3})x^{-5/3} = -\frac{1}{3x^{5/3}}$

e. $y = -\frac{3x}{2}$ $y' = \frac{d}{dx}[-\frac{3}{2}x] = -\frac{3}{2}(1) = -\frac{3}{2}$

(%i13) y:2/x;

(%o13) $\frac{2}{x}$

(%i14) diff(y,x);

(%o14) $-\frac{2}{x^2}$

(%i15) f(t):=4*t^2/5;

(%o15) $f(t):=\frac{4\,t^2}{5}$

(%i16) diff(f(t),t);

(%o16) $\frac{8\,t}{5}$

(%i17) y:2*sqrt(x);

(%o17) $2\sqrt{x}$

(%i18) diff(y,x);

(%o18) $\frac{1}{\sqrt{x}}$

(%i19) y:1/(2*x^(2/3));

(%o19) $\frac{1}{2\,x^{2/3}}$

(%i20) diff(y,x);

(%o20) $-\frac{1}{3\,x^{5/3}}$

(%i21) y:-3*x/2;

(%o21) $-\frac{3\,x}{2}$

(%i22) diff(y,x);

(%o22) $-\frac{3}{2}$

Example 6 Using Parentheses When Differentiating

Original Function	Rewrite	Differentiate	Simplify
a. $y=\frac{5}{2x^3}$.	$y=\frac{5}{2}(x^{-3})$	$y'=\frac{5}{2}(-3x^{-4})$	$y'=-\frac{15}{2x^4}$
b. $y=\frac{5}{(2x)^3}$	$y=\frac{5}{8}(x^{-3})$	$y'=\frac{5}{8}(-3x^{-4})$	$y'=-\frac{15}{8x^4}$
c. $y=\frac{7}{3x^{-2}}$	$y=\frac{7}{3}(x^2)$	$y'=\frac{7}{3}(2x)$	$y'=\frac{14x}{3}$
d. $y=\frac{7}{(3x)^{-2}}$	$y=63(x^2)$	$y'=63(2x)$	$y'=126x$

(%i1) y:5/(2*x^3);

(%o1) $\frac{5}{2\,x^3}$

(%i2) diff(y,x);

(%o2) $-\frac{15}{2\,x^4}$

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(%i7) y:5/((2*x)^3);

(%o7) $\frac{5}{8\,x^3}$

(%i4) diff(y,x);

(%o4) $-\frac{15}{8\,x^4}$

(%i5) y:7/(3*x^(-2));

(%o5) $\frac{7\,x^2}{3}$

(%i6) diff(y,x);

(%o6) $\frac{14\,x}{3}$

(%i8) y:7/((3*x)^(-2));

(%o8) $63\,x^2$

(%i9) diff(y,x);

(%o9) $126\,x$

Example 7 Using the Sum and Difference Rule

Function	Derivative
a. $f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b. $g(x) = -\frac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

(%i11) f(x):=x^3-4*x+5;

(%o11) $f(x):=x^3-4\,x+5$

(%i13) diff(f(x),x);

(%o13) $3\ x^2-4$

(%i15) g(x):=-x^4/2+3*x^3-(2*x);

(%o15) $g(x):=\frac{-x^4}{2}+3\ x^3-2\ x$

(%i17) diff(g(x),x);

(%o17) $-2\ x^3+9\ x^2-2$

Example 8 Derivatives Involving Sines and Cosines

Function	Derivative
a. $y=2\sin x$	$y'=2\cos x$
b. $y=\frac{\sin x}{2}=\frac{1}{2}\sin x$	$y'=\frac{1}{2}\cos x=\frac{\cos x}{2}$
c. $y=x+\cos x$	$y'=1-\sin x$

(%i18) y:2*sin(x);

(%o18) 2 sin(x)

(%i19) diff(y,x);

(%o19) 2 cos(x)

(%i20) y:sin(x)/2;

(%o20) $\frac{\sin(x)}{2}$

(%i21) diff(y,x);

(%o21) $\frac{\cos(x)}{2}$

(%i22) y:x+cos(x);

(%o22) $\cos(x)+x$

(%i23) diff(y,x);

(%o23) $1-\sin(x)$

Example 9 Finding Average Velocity of a Falling Object

If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function

$s=-16t^2+100$ Position function

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a.[1, 2] b.[1, 1.5] c.[1, 1.1]

Sol:

a. For the interval [1, 2], the object falls from a height of $s(1)=-16(1)^2+100=84$ feet to a height of $s(2)=-16(2)^2+100=36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t}=\frac{36-84}{2-1}=\frac{-48}{1}=-48 \text{ feet per second.}$$

b. For the interval [1, 1.5], the object falls from a height of 84 feet to a height of 64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t}=\frac{64-84}{1.5-1}=\frac{-20}{0.5}=-40 \text{ feet per second.}$$

c. For the interval [1, 1.1], the object falls from a height of 84 feet to a height of 80.64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t}=\frac{80.64-84}{1.1-1}=\frac{-3.36}{0.1}=-33.6 \text{ feet per second.}$$

Note that the average velocities are negative, indicating that the object is moving downward.

```
(%i25) s(t):=-(16*t^2)+100;
(%o25) s(t):=-16 t^2 +100

(%i26) (s(2)-s(1))/(2-1);
(%o26) -48

(%i27) (s(1.5)-s(1))/(1.5-1);
(%o27) -40.0

(%i28) (s(1.1)-s(1))/(1.1-1);
(%o28) -33.59999999999997

(%i40) fpprec:3;        (顯現位數 3 位)
(%o40) 3

(%i41) "%i28;
(%o41) -33.59999999999997

(%i42) bfloat (%o35);        (b1  表示 10 的一次)
(%o42) -3.36b1
```

Example 10 Using the Derivative to Find Velocity

At time $t=0$, a diver jumps from a platform diving board that is 32 feet above the water (see Figure 2.21). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32 \qquad \text{Position function}$$

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

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b. What is the diver's velocity at impact?

Sol:

a. To find the time t when the diver hits the water, let $s = 0$ and solve for t.

$-16t^2 + 16t + 32 = 0$	Set position function equal to 0.
$-16(t+1)(t-2) = 0$	Factor.
$t = -1 \quad or\ 2$	Solve for t.

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

b. The velocity at time t is given by the derivative $s'(t) = -32t + 16$. So, the velocity at time $t = 2$ is

$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.}$$

(%i43) factor(-(16*t^2)+16*t+32);

(%o43) $-16(t-2)(t+1)$

(%i44) solve([-(16*t^2)+16*t+32=0]);

(%o44) $[t=2, t=-1]$

(%i46) f(t):=-(16*t^2)+16*t+32;

(%o46) $f(t):=-16t^2+16t+32$

(%i50) diff(f(t),t);

(%o50) $16-32t$

(%i51) s(t):=16-32*t;

(%o51) $s(t):=16-32t$

(%i52) s(2);

(%o52) -48

Ch 2.3 Product and Quotient Rules and Higher-Order Derivatives

Example 1 Using the Product Rule

Find the derivative of $h(x)=(3x-2x^2)(5+4x)$.

Sol:

$$\begin{aligned} h'(x) &= (3x-2x^2)\frac{d}{dx}[5+4x]+(5+4x)\frac{d}{dx}[3x-2x^2] \quad \text{Apply Product Rule.} \\ &= (3x-2x^2)(4)+(5+4x)(3-4x) \\ &= (12-8x^2)+(15-8x-16x^2) \\ &= -24x^2+4x+15 \end{aligned}$$

(%i4) h:(3*x-2*x^2)*(5+4*x);

(%o4) $(4\,x+5)(3\,x-2\,x^2)$

(%i5) expand(%o4);

(%o5) $-8\,x^3+2\,x^2+15\,x$

(%i7) diff(%o5,x);

(%o7) $-24\,x^2+4\,x+15$

Example 2 Using the Product Rule

Find the derivative of $y=3x^2\sin x$

Sol:

$$\begin{aligned} \frac{d}{dx}[3x^2\sin x] &= 3x^2\frac{d}{dx}[\sin x]+\sin x\frac{d}{dx}[3x^2] \\ &= 3x^2\cos x+(\sin x)(6x) \\ &= 3x^2\cos x+6x\sin x \\ &= 3x(x\cos x+2\sin x) \end{aligned}$$

(%i1) f(x):=3*x^2*sin(x);

(%o1) $f(x):=3\,x^2\,\sin(x)$

(%i2) diff(f(x),x);

(%o2) $6\,x\sin(x)+3\,x^2\cos(x)$

Example 3 Using the Product Rule

Find the derivative of $y = 2x\cos x - 2\sin x$.

Sol:

$$\frac{dy}{dx} = (2x)(\frac{d}{dx}[\cos x]) + (\cos x)(\frac{d}{dx}[2x]) - 2\frac{d}{dx}[\sin x]$$
$$= (2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$$
$$= -2x\sin x$$

(%i3) g(x):=2*x*cos(x)-2*sin(x);

(%o3) $g(x):=2\,x\cos(x)-2\sin(x)$

(%i4) diff(g(x),x);

(%o4) $-2\,x\sin(x)$

Example 4 Using the Quotient Rule

Find the derivative of $y = \frac{5x-2}{x^2+1}$

Sol:

$$\frac{d}{dx}[\frac{5x-2}{x^2+1}] = \frac{(x^2+1)\frac{d}{dx}[5x-2]-(5x-2)\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$$
$$= \frac{(x^2+1)(5)-(5x-2)(2x)}{(x^2+1)^2}$$
$$= \frac{(5x^2+5)-(10x^2-4x)}{(x^2+1)^2}$$
$$= \frac{-5x^2+4x+5}{(x^2+1)^2}$$

(%i5) h(x):=(5*x-2)/(x^2+1);

(%o5) $h(x):=\frac{5x-2}{x^2+1}$

(%i6) diff(h(x),x);

(%o6) $\frac{5}{x^2+1}-\frac{2x(5x-2)}{(x^2+1)^2}$

Example 5 Rewriting Before Differentiating

Find an equation of the tangent line to the graph of $f(x)=\frac{3-(1/x)}{x+5}$ at (-1, 1).

Sol:

Begin by rewriting the function.

$$f(x)=\frac{3-(1/x)}{x+5}$$

$$=\frac{x(3-\frac{1}{x})}{x(x+5)}$$

$$=\frac{3x-1}{x^2+5x}$$

$$f'(x)=\frac{(x^2+5x)(3)-(3x-1)(2x+5)}{(x^2+5x)^2}$$

$$=\frac{(3x^2+15x)-(6x^2+13x-5)}{(x^2+5x)^2}$$

$$=\frac{-3x^2+2x+5}{(x^2+5x)^2}$$

To find the slope at (-1, 1), evaluate $f'(-1)$.

$$f'(-1)=0$$

Then, using the point-slope form of the equation of a line, you can determine that the equation of the tangent line at (-1, 1) is $y=1$. See Figure 2.23

Example 6 Using the Constant Multiple Rule

Original Function	Rewrite	Differentiate	Simplify
a. $y=\dfrac{x^2+3x}{6}$.	$y=\dfrac{1}{6}(x^2+3x)$	$y'=\dfrac{1}{6}(2x+3)$	$y'=\dfrac{2x+3}{6}$
b. $y=\dfrac{5x^4}{8}$	$y=\dfrac{5}{8}x^4$	$y'=\dfrac{5}{8}(4x^3)$	$y'=\dfrac{5}{2}x^3$
c. $y=\dfrac{-3(3x-2x^2)}{7x}$	$y=-\dfrac{3}{7}(3-2x)$	$y'=-\dfrac{3}{7}(-2)$	$y'=\dfrac{6}{7}$
d. $y=\dfrac{9}{5x^2}$	$y=\dfrac{9}{5}(x^{-2})$	$y'=\dfrac{9}{5}(-2x^{-3})$	$y'=-\dfrac{18}{5x^3}$

(%i11) y:(x^2+3*x)/6;

(%o11) $\dfrac{x^2+3x}{6}$

(%i12) diff(y,x);

(%o12) $\dfrac{2x+3}{6}$

(%i13) y:5*x^4/8;

(%o13) $\dfrac{5x^4}{8}$

(%i14) diff(y,x);

(%o14) $\dfrac{5x^3}{2}$

(%i15) y:-3*(3*x-2*x^2)/(7*x);

(%o15) $-\dfrac{3(3x-2x^2)}{7x}$

(%i17) y:-3*(3-2*x)/7;

(%o17) $-\frac{3(3-2x)}{7}$

(%i18) diff(y,x);

(%o18) $\frac{6}{7}$

(%i19) y:9/(5*x^2);

(%o19) $\frac{9}{5x^2}$

(%i20) diff(y,x);

(%o20) $-\frac{18}{5x^3}$

Example 7 Proof of the Power Rule (Negative Integer Exponents)

If n is a negative integer, there exists a positive integer k such that $n=-k$. So, by the Quotient Rule, you can write

$$\frac{d}{dx}[x^n]=\frac{d}{dx}[\frac{1}{x^k}]$$

$$=\frac{x^k(0)-(1)(kx^{k-1})}{(x^k)^2} \qquad \text{Quotient Rule and Power Rule}$$

$$=\frac{0-kx^{k-1}}{x^{2k}}$$

$$=-kx^{-k-1}$$

$$=nx^{n-1} \qquad n=-k$$

So, the Power Rule

$$D_x[x^n]=nx^{n-1} \qquad \text{Power Rule}$$

is valid for any integer. In Exercise 75 in Section 2.5, you are asked to prove the case for which n is any rational number.

http://www.npue.edu.tw/academic/math/index.htm

Example 8

Function	Derivative
a. $y = x - \tan x$	$\frac{dy}{dx} = 1 - \sec^2 x$
b. $y = x\sec x$	$y' = x(\sec x \tan x) + (\sec x)(1)$ $= (\sec x)(1 + x\tan x)$

(%i21) y:x-tan(x);

(%o21) x-tan(x)

(%i22) diff(y,x);

(%o22) $1-\sec(x)^2$

(%i23) y:x*sec(x);

(%o23) x sec(x)

(%i24) diff(y,x);

(%o24) x sec(x) tan(x)+sec(x)

Example 9

Differentiate both form of $y = \frac{1-\cos x}{\sin x} = \csc x - \cot x$.

Sol:

First form : $y = \frac{1-\cos x}{\sin x}$

$$y' = \frac{(\sin x)(\sin x) - (1-\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x}$$

$$= \frac{1-\cos x}{\sin^2 x}$$

Second form : $y = \csc x - \cot x$

$$y' = -\csc x \cot x + \csc^2 x$$

To show that the two derivatives are equal, you can write

$$\frac{1-\cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - (\frac{1}{\sin x})(\frac{\cos x}{\sin x})$$
$$= \csc^2 x - \csc x \cot x$$

(%i25) y:(1-cos(x))/sin(x);

(%o25) $\frac{1-\cos(x)}{\sin(x)}$

(%i26) diff(y,x);

(%o26) $1-\frac{(1-\cos(x))\cos(x)}{\sin(x)^2}$

(%i56) trigsimp(%o26);

(%o56) $-\frac{\cos(x)-1}{\sin(x)^2}$

(%i27) y:csc(x)-cot(x);

(%o27) $\csc(x)-\cot(x)$

(%i28) diff(y,x);

(%o28) $\csc(x)^2-\cot(x)\csc(x)$

(%i57) trigsimp(%o28);

(%o57) $-\frac{\cos(x)-1}{\sin(x)^2}$

Example 10

Because the moon has no atmosphere, a falling object on the moon encounters no air resistance. In 1971, astronaut David Scott demonstrated that a feather and a hammer fall at the same rate on the moon. The position function for each of these falling objects is given by

$$s(t) = -0.81t^2 + 2$$

where $s(t)$ is the height in meters and t is the time in seconds. What is the ratio of Earth's gravitational force to the moon's?

Sol:

To find the acceleration, differentiate the position function twice.

$s(t) = -0.81t^2 + 2$ Position function

$s'(t) = -1.62t$ Velocity function

$s''(t) = -1.62$ Acceleration function

So, the acceleration due to gravity on the moon is -1.62 meters per second per second. Because the acceleration due to gravity on Earth is -9.8 meters per second per second, the ratio of Earth's gravitational force to the moon's is

$$\frac{Earth's\ \ gravitational\ \ force}{Moon's\ \ gravitational\ \ force} = \frac{-9.8}{-1.62} \approx 6.05$$

(%i34) s(t):=-0.81*t^2+2;

(%o34) $s(t):=(-0.81)\,t^2+2$

(%i35) diff(s(t),t);

(%o35) $-1.62\ t$

(%i37) diff(s(t),t,2);

(%o37) -1.62

Ch 2.4 The Chain Rule

Example 1 The derivative of a Composite Function

A set of gears is constructed, as shown in Figure 2.24, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles. Find $dy/du, du/dx$, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Sol:

Because the circumference of the gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$\frac{dy}{dx} =$ (Rate of change of first axle with respect to second axle) · (Rate of change of second axle with respect to third axle)

$$= \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6$$

= (Rate of change of first axle with respect to third axle)

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

Example 2 Decomposition of a Composite Function

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \frac{1}{x+1}$	$u = x+1$	$y = \frac{1}{u}$
b. $y = \sin 2x$	$u = 2x$	$y = \sin u$

c. $y = \sqrt{3x^2 - x + 1}$ $\quad u = 3x^2 - x + 1$ $\quad y = \sqrt{u}$

d. $y = \tan^2 x$ $\quad u = \tan x$ $\quad y = u^2$

Example 3 Using the Chain Rule

Find dy/dx for $y = (x^2 + 1)^3$.

Sol:

For this function, you can consider the inside function to be $u = x^2 + 1$ By the Chain Rule, you obtain

$$\frac{dy}{dx} = 3(x^2+1)^2(2x) = 6x(x^2+1)^2$$

$$\downarrow \qquad \downarrow$$

$$\frac{dy}{du} \qquad \frac{du}{dx}$$

(%i38) y:(x^2+1)^3;

(%o38) $(x^2+1)^3$

(%i39) diff(y,x);

(%o39) $6\,x\,(x^2+1)^2$

Example 4 Applying the General Power Rule

Find the derivative of $f(x) = (3x - 2x^2)^3$

Sol:

Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$f'(x) = 3(3x-2x^2)^2 \frac{d}{dx}[3x - 2x^2]$$ Apply General Power Rule.

$$= 3(3x-2x^2)^2(3-4x).$$ Differentiate $3x - 2x^2$.

http://www.npue.edu.tw/academic/math/index.htm

(%i40) f(x):=(3*x-2*x^2)^3;

$$\text{(\%o40)}\quad f(x):=(3x-2x^2)^3$$

(%i41) diff(f(x),x);

$$\text{(\%o41)}\quad 3(3-4x)(3x-2x^2)^2$$

Example 5 Differentiating Functions Involving Radicals

Find all points on the graph of $f(x)=\sqrt[3]{(x^2-1)^2}$ for which $f'(x)=0$ and those for which $f'(x)$ does not exist.

Sol:

Begin by rewriting the function as

$$f(x)=(x^2-1)^{2/3}$$

Then, applying the General Power Rule (with $u=x^2-1$) produces

$$f'(x)=\frac{2}{3}(x^2-1)^{-1/3}(2x) \qquad \text{Apply General Power Rule.}$$

$$=\frac{4x}{3\sqrt[3]{x^2-1}} \qquad \text{Write in radical form.}$$

So, $f'(x)=0$ when $x=0$ and $f'(x)$ does not exist when $x=\pm 1$, as shown in Figure 2.25.

Example 6 Differentiating Quotients with Constant Numerators

Differentiate $g(t)=\dfrac{-7}{(2t-3)^2}$

Sol:

Begin by rewriting the function as

$$g(t)=-7(2t-3)^{-2}$$

Then, applying the General Power Rule produces

$$g'(t)=(-7)(-2)(2t-3)^{-3}(2) \qquad \text{Apply General Power Rule.}$$

$$=28(2t-3)^{-3} \qquad \text{Simplify.}$$

$$=\frac{28}{(2t-3)^3} \qquad \text{Write with positive exponent.}$$

(%i42) g(t):=-7/(2*t-3)^2;

(%o42) $g(t):=\frac{-7}{(2t-3)^2}$

(%i43) diff(g(t),t);

(%o43) $\frac{28}{(2t-3)^3}$

Example 7 Simplifying by Factoring Out the Least Powers

$$f(x)=x^2\sqrt{1-x^2}$$ Original function

$$=x^2(1-x^2)^{1/2}$$ Rewrite

$$f'(x)=x^2\frac{d}{dx}[(1-x^2)^{1/2}]+(1-x^2)^{1/2}\frac{d}{dx}[x^2]$$ Product Rule

$$=x^2[\frac{1}{2}(1-x^2)^{-1/2}(-2x)]+(1-x^2)^{1/2}(2x)$$ General Power Rule

$$=-x^3(1-x^2)^{-1/2}+2x(1-x^2)^{1/2}$$ Simplify

$$=x(1-x^2)^{-1/2}[-x^2(1)+2(1-x^2)]$$ Factor

$$=\frac{x(2-3x^2)}{\sqrt{1-x^2}}$$ Simplify

(%i44) f(x):=x^2*sqrt(1-x^2);

(%o44) $f(x):=x^2\sqrt{1-x^2}$

(%i45) diff(f(x),x);

(%o45) $2x\sqrt{1-x^2}-\frac{x^3}{\sqrt{1-x^2}}$

(%i47) factor(%o46);

(%o47) $-\frac{x(3x^2-2)}{\sqrt{1-x^2}}$

Example 8 Simplifying the Derivative of a Quotient

$$f(x)=\frac{x}{\sqrt[3]{x^2+4}}$$ Original function

$$=\frac{x}{(x^2+4)^{1/3}}$$ Rewrite

$$f'(x)=\frac{(x^2+4)^{1/3}(1)-x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}}$$ Quotient Rule

$$=\frac{1}{3}(x^2+4)^{-2/3}[\frac{3(x^2+4)-(2x^2)(1)}{(x^2+4)^{2/3}}]$$ Factor

$$=\frac{x^2+12}{3(x^2+4)^{4/3}}$$ Simplify

(%i53) f(x):=x/(x^2+4)^(1/3);

(%o53) $f(x):=\frac{x}{(x^2+4)^{1/3}}$

(%i54) diff(f(x),x);

(%o54) $\frac{1}{(x^2+4)^{1/3}}-\frac{2x^2}{3(x^2+4)^{4/3}}$

(%i55) factor(%o54);

(%o55) $\frac{x^2+12}{3(x^2+4)^{4/3}}$

Example 9 Simplifying the Derivative of a Power

$$y=(\frac{3x-1}{x^2+3})^2$$ Original function

$$y'=2(\frac{3x-1}{x^2+3})\frac{d}{dx}[\frac{3x-1}{x^2+3}]$$ General Power Rule

$$=[\frac{2(3x-1)}{x^2+3}][\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}]$$ Quotient Rule

$$= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3}$$ Multiply

$$= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$$ Simplify

(%i58) y:((3*x-1)/(x^2+3))^2;

(%o58) $\frac{(3x-1)^2}{(x^2+3)^2}$

(%i59) diff(y,x);

(%o59) $\frac{6(3x-1)}{(x^2+3)^2} - \frac{4x(3x-1)^2}{(x^2+3)^3}$

(%i60) factor(%o59);

(%o60) $-\frac{2(3x-1)(3x^2-2x-9)}{(x^2+3)^3}$

Example 10 Applying the Chain Rule to Trigonometric Functions

a. $y = \sin 2x$ $y' = \cos 2x \frac{d}{dx}[2x] = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos(x-1)$ $y' = -\sin(x-1)$

c. $y = \tan 3x$ $y' = 3\sec^2 3x$

(%i61) y:sin(2*x);

(%o61) sin(2 x)

(%i62) diff(y,x);

(%o62) 2 cos(2 x)

(%i63) y:cos(x-1);

(%o63) cos(x−1)

(%i64) diff(y,x);

(%o64) −sin(x−1)

(%i65) y:tan(3*x);

(%o65) tan(3 x)

(%i66) diff(y,x);

(%o66) $3\sec(3x)^2$

Example 11 Parentheses and Trigonometric Functions

a. $y=\cos 3x^2=\cos(3x^2)$ $y'=(-\sin 3x^2)(6x)=-6x\sin 3x^2$

b. $y=(\cos 3)x^2$ $y'=(-\cos 3)(2x)=2x\cos 3$

c. $y=\cos(3x)^2=\cos(9x^2)$ $y'=(-\sin 9x^2)(18x)=-18x\sin 9x^2$

d. $y=\cos^2 x=(\cos x)^2$ $y'=2(\cos x)(-\sin x)=-2\cos x\sin x$

e. $y=\sqrt{\cos x}=(\cos x)^{1/2}$ $y'=\frac{1}{2}(\cos x)^{-1/2}(-\sin x)=-\frac{\sin x}{2\sqrt{\cos x}}$

(%i1) y:cos(3*x^2);

(%o1) $\cos(3x^2)$

(%i2) diff(y,x);

(%o2) $-6x\sin(3x^2)$

(%i1) y:(cos(3))*x^2;

(%o1) $\cos(3)x^2$

(%i2) diff(y,x);

(%o2) 2 cos(3) x

Example 12　Repeated Application of the Chain Rule

$f(t)=\sin^3 4t$　Original function

$=(\sin 4t)^3$　Rewrite

$f'(t)=3(\sin 4t)^2\dfrac{d}{dt}[\sin 4t]$　Apply Chain Rule once.

$=3(\sin 4t)^2(\cos 4t)\dfrac{d}{dt}[4t]$　Apply Chain Rule a second time.

$=3(\sin 4t)^2(\cos 4t)(4)$

$=12\sin^2 4t\cos 4t$　Simplify.

Example 13　Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of

$$f(x)=2\sin x+\cos 2x$$

at the point $(\pi,1)$, as shown in Figure 2.26. Then determine all value of x in the interval $(0,2\pi)$ at which the graph of f has a horizontal tangent.

Sol:

Begin by finding $f'(x)$

$f(x)=2\sin x+\cos 2x$　Write original function.

$f'(x)=2\cos x+(-\sin 2x)(2)$　Apply Chain Rule to $\cos 2x$

$=2\cos x-2\sin 2x$　Simplify.

To find the equation of the tangent line at $(\pi,1)$, evaluate $f'(\pi)$.

$f'(\pi)=2\cos\pi-2\sin 2\pi$　Substitute.

$=-2$　Slope of graph at $(\pi,1)$

Now, using the point-slope from of the equation of a line, you can write

$y-y_1=m(x-x_1)$　Point-slope form

$y-1=-2(x-\pi)$　Substitute for y_1, m, and x_1

$y=1-2x+2\pi$　Equation of tangent line at $(\pi,1)$

You can then determine that $f'(x)=0$ when $x=\dfrac{\pi}{6},\dfrac{\pi}{2},\dfrac{5\pi}{6}$, and $\dfrac{3\pi}{2}$. So, f has a horizontal tangent at $x=\dfrac{\pi}{6},\dfrac{\pi}{2},\dfrac{5\pi}{6}$, and $\dfrac{3\pi}{2}$.

Ch 2.5 Implicit Differentiation

Example 1 Differentiating with Respect to x

a. $\frac{d}{dx}[x^3]=3x^2$ Variables agree : use Simple Power Rule

b. $\frac{d}{dx}[y^3]=3y^2\frac{dy}{dx}$ Variables agree : use Chain Rule

c. $\frac{d}{dx}[x+3y]=1+3\frac{dy}{dx}$ Chain Rule: $\frac{d}{dx}[3y]=3y'$

d. $\frac{d}{dx}[xy^2]=x\frac{d}{dx}[y^2]+y^2\frac{d}{dx}[x]$ Product Rule

$=x\left(2y\frac{dy}{dx}\right)+y^2(1)$ Chain Rule

$=2xy\frac{dy}{dx}+y^2$ Simplify

Example 2 Implicit Differentiation

Find dy/dx given that $y^3+y^2-5y-x^2=-4$

Sol:

1. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}[y^3+y^2-5y-x^2]=\frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3]+\frac{d}{dx}[y^2]-\frac{d}{dx}[5y]-\frac{d}{dx}[x^2]=\frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx}+2y\frac{dy}{dx}-5\frac{dy}{dx}-2x=0$$

2. Collect the dy/dx terms on the left side of the equation and move all other terms to the right side of the equation.

$$3y^2\frac{dy}{dx}+2y\frac{dy}{dx}-5\frac{dy}{dx}=2x$$

3. Factor dy/dx out of the left side of the equation.

$$\frac{dy}{dx}(3y^2 + 2y - 5) = 2x$$

4. Solve for dy/dx by dividing by $(3y^2 + 2y - 5)$.

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

Example 3 Representing a Graph by Differentiable Functions

If possible, represent y as a differentiable function of x.

a. $x^2 + y^2 = 0$ b. $x^2 + y^2 = 1$ c. $x + y^2 = 1$

Sol:

a. The graph of this equation is a single point. So, it does not define y as a differentiable function of x. See Figure 2.28(a).

b. The graph of this equation is the unit circle, centered at (0, 0). The upper semicircle is given by the differentiable function

$$y = \sqrt{1 - x^2}, \qquad -1 < x < 1$$

and the lower semicircle is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \qquad -1 < x < 1$$

At the points (-1, 0) and (1, 0), the slope of the graph is undefined. See Figure 2.28(b).

c. The upper half of this parabola is given by the differentiable function

$$y = \sqrt{1 - x^2}, \qquad x < 1$$

and the lower half of this parabola is given by the differentiable function

$$y = -\sqrt{1 - x^2}, \qquad x < 1$$

At the point (1, 0), the slope of the graph is undefined. See Figure 2.28(c).

Example 4 Finding the Slope of a Graph Implicitly

Determine the slope of the tangent line to the graph of

$$x^2 + 4y^2 = 4$$

at the point $(\sqrt{2}, -1/\sqrt{2})$. See Figure 2.29.

Sol:

$$x^2 + 4y^2 = 4$$

$$2x + 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y}$$

So, at $(\sqrt{2}, -1/\sqrt{2})$., the slope is

$$\frac{dy}{dx} = \frac{-\sqrt{2}}{-4/\sqrt{2}} = \frac{1}{2}$$

Example 5 Finding the Slope of a Graph Implicitly

Determine the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at the point (3, 1).

Sol:

$$\frac{d}{dx}[3(x^2 + y^2)^2] = \frac{d}{dx}[100xy]$$

$$3(2)(x^2 + y^2)(2x + 2y\frac{dy}{dx}) = 100[x\frac{dy}{dx} + y(1)]$$

$$12y(x^2 + y^2)\frac{dy}{dx} - 100\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$[12y(x^2 + y^2) - 100x]\frac{dy}{dx} = 100y - 12x(x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{100y - 12x(x^2 + y^2)}{12y(x^2 + y^2) - 100x}$$

$$= \frac{25y - 3x(x^2 + y^2)}{-25x + 3y(x^2 + y^2)}$$

At the point (3, 1), the slope of the graph is

$$\frac{dy}{dx} = \frac{25(1) - 3(3)(3^2 + 1^2)}{-25(3) + 3(1)(3^2 + 1^2)} = \frac{25 - 90}{-75 + 30} = \frac{-65}{-45} = \frac{13}{9}$$

as shown in Figure 2.30. This graph is called a lemniscates.

Example 6 Determining a Differentiable Function

Find dy/dx implicitly for the equation $\sin y = x$.Then find the largest interval of the form $-a < y < a$ on which y is a differentiable function of x (see Figure 2.31).

Sol:

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

The largest interval about the origin for which y is a differentiable function of x is $-\pi/2 < y < \pi/2$. To see this, note that $\cos y$ is positive for all y in this interval and is 0 at the endpoints. If you restrict y to the interval $-\pi/2 < y < \pi/2$. you should be able to write dy/dx explicitly as a function of x. To do this, you can use

$$\cos y = \sqrt{1-\sin^2 y}$$

$$= \sqrt{1-x^2}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

and conclude that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Example 7 Finding the Second Derivative Implicitly

Given $x^2 + y^2 = 25$, find $\frac{d^2y}{dx^2}$

Sol:

Differentiating each term with respect to x produces

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Differentiating a second time with respect to x yields.

$$\frac{d^2y}{dx^2} = -\frac{(y)(1)-(x)(dy/dx)}{y^2}$$ Quotient Rule

$$= -\frac{y-(x)(-x/y)}{y^2}$$ Substitute $-x/y$ for $\frac{dy}{dx}$

$$= -\frac{y^2+x^2}{y^3}$$ Simplify

$$= -\frac{25}{y^3}$$ Substitute 25 for $x^2 + y^2$

Example 8 Finding a Tangent Line to a Graph

Find the tangent line to the graph given by $x^2(x^2+y^2)=y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$, as shown in Figure 2.32.

Sol:

By rewriting and differentiating implicitly, you obtain

$$x^4+x^2y^2-y^2=0$$

$$4x^3+x^2(2y\frac{dy}{dx})+2xy^2-2y\frac{dy}{dx}=0$$

$$2y(x^2-1)\frac{dy}{dx}=-2x(2x^2+y^2)$$

$$\frac{dy}{dx}=\frac{x(2x^2+y^2)}{y(1-x^2)}$$

At the point $(\sqrt{2}/2, \sqrt{2}/2)$, the slope is

$$\frac{dy}{dx}=\frac{(\sqrt{2}/2)[2(1/2)+(1/2)]}{(\sqrt{2}/2)[1-(1/2)]}=\frac{3/2}{1/2}=3$$

and the equation of the tangent line at this point is

$$y-\frac{\sqrt{2}}{2}=3(x-\frac{\sqrt{2}}{2})$$

$$y=3x-\sqrt{2}$$