12.1 DOUBLE INTEGRALS

Example1 Find the double Riemann sum

$$\sum_{D_1}\sum x^2 y\Delta x\Delta y,$$

Where D_1 is the square

$$0 \le x \le 1, \qquad 0 \le y \le 1,$$

and

$$\Delta x = \frac{1}{4}, \qquad \Delta y = \frac{1}{5}.$$

The partition of D_1 is shown in Figure 12.1.10 and the values of $x^2 y$ at the partition points are shown in the table.

$x^2 y$	$y_0 = 0$	$y_1 = \frac{1}{5}$	$y_2 = \frac{2}{5}$	$y_3 = \frac{3}{5}$	$y_4 = \frac{4}{5}$
$x_0 = 0$	0	0	0	0	0
$x_1 = \frac{1}{4}$	0	$\frac{1}{80}$	$\frac{2}{80}$	$\frac{3}{80}$	$\frac{4}{80}$
$x_2 = \frac{1}{2}$	0	$\frac{4}{80}$	$\frac{8}{80}$	$\frac{12}{80}$	$\frac{16}{80}$
$x_3 = \frac{3}{4}$	0	$\frac{9}{80}$	$\frac{18}{80}$	$\frac{27}{80}$	$\frac{36}{80}$

The double Riemann sum is

$$\sum_{D_1}\sum x^2 y\Delta x\Delta y$$

$$=(1+2+3+4+4+8+12+16+9+18+27+36)\frac{1}{80}\cdot\frac{1}{4}\cdot\frac{1}{5}=0.0875.$$

A similar computation with $\Delta x = \frac{1}{10}$, $\Delta y = \frac{1}{10}$ gives

$$\sum_{D_1}\sum x^2 y\Delta x\Delta y = 0.12825.$$

(%i10) solve[(1+2+3+4+4+8+12+16+9+18+27+36)*(1/80)*(1/4)*(1/5)];

(%o10) $solve_{7/80}$

(%i11) float(%), numer;

(%o11) $solve_{0.0875}$

Example 2 Find the double Riemann sum

$$\sum_{D_2}\sum x^2 y\Delta x\Delta y$$

Where D_2 is the region

$$0 \le x \le 1, \qquad x^2 \le y \le \sqrt{x}$$

and

$$\Delta x = \frac{1}{4}, \qquad \Delta y = \frac{1}{5}.$$

The circumscribed rectangle of D_2 is the unit square. The partition and D_2 are shown in Figure 12.1.11 and the partition points which actually belong to D_2 are circled. The table shows the values of $x^2 y$ at the partition points which belong to D_2 .It is a part of the table from Example 1.

x^2y	$y_0=0$	$y_1=\frac{1}{5}$	$y_2=\frac{2}{5}$	$y_3=\frac{3}{5}$	$y_4=\frac{4}{5}$
$x_0=0$	0				
$x_1=\frac{1}{4}$		$\frac{1}{80}$	$\frac{2}{80}$		
$x_2=\frac{1}{2}$			$\frac{8}{80}$	$\frac{12}{80}$	
$x_3=\frac{3}{4}$				$\frac{27}{80}$	$\frac{36}{80}$

The double Riemann sum is

$$\sum_{D_2}\sum x^2 y\Delta x\Delta y$$

$$=(\frac{1}{80}+\frac{2}{80}+\frac{8}{80}+\frac{12}{80}+\frac{27}{80}+\frac{36}{80})\frac{1}{4}\cdot\frac{1}{5}=\frac{86}{80\cdot 4\cdot 5}=0.05375.$$

A similar computation with $\Delta x=\frac{1}{10}$, $\Delta y=\frac{1}{10}$ gives

$$\sum_{D_2}\sum x^2 y\Delta x\Delta y=0.04881.$$

(%i18) (1+2+8+12+27+36)*(1/80)*(1/4)*(1/5);

(%o18) $\frac{43}{800}$

(%i19) float(%), numer;

(%o19) 0.05375

12.2 ITERATED INTEGRALS

Example 1 Evaluate $\iint_{D_1} x^2 y \, dA$

Where D_1 is the unit square

$$0 \le x \le 1, \qquad 0 \le y \le 1$$

The limits of the outside integral are given by $0 \le x \le 1$, and those of the inside integral are given by $0 \le y \le 1$ .The iterated integral is thus

$$\iint_{D_1} x^2 y \, dA = \int_0^1 \int_0^1 x^2 y \, dy \, dx .$$

The inside integral is

$$\int_0^1 x^2 y \, dy = \frac{1}{2} x^2 y^2 \Big]_{y=0}^{y=1} = \frac{1}{2} x^2 .$$

Then $$\iint_{D_1} x^2 y \, dA = \int_0^1 \frac{1}{2} x^2 \, dx = \frac{1}{6} x^3 \Big]_0^1 = \frac{1}{6} \sim 0.16667.$$

Since D_1 is a rectangle we may also integrate in the other order, and should get the same answer.

$$\iint_{D_1} x^2 y \, dA = \int_0^1 \int_0^1 x^2 y \, dx \, dy$$

$$\int_0^1 x^2 y \, dy = \frac{1}{3} x^3 y \Big]_0^1 = \frac{1}{3} y.$$

$$\iint_{D_1} x^2 y \, dA = \int_0^1 \frac{1}{3} y \, dy = \frac{1}{6} y^2 \Big]_0^1 = \frac{1}{6} \sim 0.16667.$$

The Riemann sums in Section 12.1 were 0.0875, 0.12825.

```
(%i2) integrate((x^2)*y, y, 0, 1);
```

(%o2) $\frac{x^2}{2}$

(%i3) integrate((x^2)/2, x, 0, 1);

(%o3) $\frac{1}{6}$

Example 2 Evaluate $\iint_{D_2} x^2 y\, dA$ where D_2 is the region in Figure 12.2.2:

$$0 \le x \le 1, \qquad x^2 \le y \le \sqrt{x}$$

The limits on the outside integral are given by $0 \le x \le 1$, and those on the inside integral by $x^2 \le y \le \sqrt{x}$, so the iterated integral is

$$\iint_{D_2} x^2 y\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y\, dy\, dx.$$

$$\int_{x^2}^{\sqrt{x}} x^2 y\, dy = \frac{1}{2} x^2 y^2 \Big]_{y=x^2}^{y=\sqrt{x}} = \frac{1}{2} x^3 - \frac{1}{2} x^6.$$

$$\iint_{D_2} x^2 y\, dA = \int_0^1 (\frac{1}{2} x^3 - \frac{1}{2} x^6)\, dx = \frac{1}{8} x^4 - \frac{1}{14} x^7 \Big]_0^1$$

$$= \frac{3}{56} \sim 0.05357.$$

The Riemann sums in Section 12.1 were 0.05357, 0.04881.

In many applications the region D is given verbally, and part of the problem is to find inequalities which describe D.

(%i12) assume(x>0);

(%o12) [x>0]

(%i13) integrate(x^2*y, y, x^2, sqrt((x)));

(%o13) $x^2\left(\frac{x}{2}-\frac{x^4}{2}\right)$

(%i14) integrate(%o13, x, 0, 1);

(%o14) $\frac{3}{56}$

Example 3 Let D be the region bounded by the curve $xy = 1$ and the line $y = \frac{5}{2} - x$. Find inequalities which describe D, and write down an iterated integral equal to $\iint_D f(x, y)\, dA$.

Step 1 Sketch the region D as in Figure 12.2.3

Step 2 The line and curve intersect where

$$x(\frac{5}{2} - x) = 1,$$

$$x^2 - \frac{5}{2}x + 1 = 0,$$

$$(x - \frac{1}{2})(x - 2) = 0,$$

$$x = \frac{1}{2}, \qquad x = 2$$

For $\frac{1}{2} \le x \le 2$, the curve $y = \frac{1}{x}$ is below the line $y = \frac{5}{2} - x$. Therefore D is the region

$$\frac{1}{2} \le x \le 2, \qquad \frac{1}{x} \le y \le \frac{5}{2} - x$$

Step 3 The inequalities for give the limits of the outside integral, and those for y give the limits of the inside integral. Thus

$$\iint_D f(x, y)\, dA = \int_{1/2}^{2} \int_{1/x}^{(5/2)-x} f(x, y)\, dy\, dx$$

Example 4 Find the volume of the solid bounded by the surfaces

$z=0,\ z=y-x^2,\ y=1.$

Step 1 Sketch the solid and the region D, as in Figure 12.2.4.

Step 2 Find the inequalities describing the region D

This is the hardest step, and gives us the limits of integration. The surfaces $z=0$ and $z=y-x^2$ intersect at the curve $y=x^2$. We see from the figure that D is the region between the curves $y=x^2$ and $y=1$, so D is given by

$$-1\le x\le 1,\qquad x^2\le y\le 1.$$

Step 3 Set up the iterated integral and evaluate it.

$$V=\iint_D y-x^2\ dA=\int_{-1}^{1}\int_{x^2}^{1} y-x^2\ dy\,dx.$$

$$\int_{x^2}^{1} y-x^2\ dy=\frac{1}{2}y^2-x^2y\Big]_{x^2}^{1}$$

$$=(\frac{1}{2}\cdot 1^2-x^2\cdot 1)-(\frac{1}{2}(x^2)^2-x^2\cdot x^2)$$

$$=\frac{1}{2}-x^2+\frac{1}{2}x^4.$$

$$V=\int_{-1}^{1}\frac{1}{2}-x^2+\frac{1}{2}x^4\ dx=\frac{16}{30}$$

(%i20) assume(x>-1);

(%o20) [redundant]

(%i21) assume(x<1);

(%o21) [x<1]

(%i23) integrate(y-x^2, y, x^2, 1);

(%o23) $\frac{x^4}{2} - \frac{2x^2 - 1}{2}$

(%i24) integrate(%o23, x, -1, 1);

(%o24) $\frac{8}{15}$

Example 5 Let D be the region bounded by the curves

$$x = y^2, \qquad x = y + 2.$$

Evaluate the double integral $\iint_D xy\, dA$.

Step 1 The region D is sketched in Figure 12.2.7.

Step 2 Find inequalities for D .To do this we must find the points where the curves

$$x = y^2, \qquad x = y + 2.$$

intersect. Solving for y and then x, we see that they intersect at

(1,-1), (4, 2).

We see from the figure that D is a region in either the (x, y) plane or the (y, x) plane. However, the boundary curves are simpler in the (y, x) plane.

D is the region

$$-1 \le y \le 2, \qquad y^2 \le x \le y + 2.$$

Step 3 Set up the iterated integral and evaluate.

$$\iint_D xy\, dA = \int_{-1}^{2}\int_{y^2}^{y+2} xy\, dx\, dy.$$

$$\int_{y^2}^{y+2} xy\, dx = \frac{1}{2}x^2 y\Big]_{y^2}^{y+2}$$

$$= \frac{1}{2}(y+2)^2 y - \frac{1}{2}(y^2)^2 y$$

$$= \frac{1}{2}y^3 + 2y^2 + 2y - \frac{1}{2}y^5.$$

$$\iint_D xy\, dA = \int_{-1}^{2} \frac{1}{2}y^3 + 2y^2 + 2y - \frac{1}{2}y^5\, dy = \frac{135}{24}.$$

(%i25) assume(y>-1);

(%o25) $[y>-1]$

(%i26) assume(y<2);

(%o26) $[y<2]$

(%i27) integrate(x*y, x, y^2, y+2);

(%o27) $y\left(\frac{y^2+4\,y+4}{2}-\frac{y^4}{2}\right)$

(%i28) integrate(%o27, y, -1, 2);

(%o28) $\frac{45}{8}$

12.3 INFINITE SUM THEOREM AND VOLUME

Example 1 Find the volume of the solid

$$0 \le x \le 1, \qquad 0 \le y \le x, \qquad x + y \le z \le e^{x+y}.$$

Step 1 D is the triangle shown in Figure 12.3.6

Step 2 D is the region $0 \le x \le 1$, $0 \le y \le x$.

Step 3
$$V = \iint_D e^{x+y} - (x+y)\, dA$$

$$= \int_0^1 \int_0^x e^{x+y} - (x+y)\, dy\, dx.$$

$$\int_0^x e^{x+y} - (x+y)\, dy = e^{x+y} - xy - \frac{1}{2}y^2 \Big]_0^x = e^{2x} - e^x - \frac{3}{2}x^2.$$

$$V = \int_0^1 e^{2x} - e^x - \frac{3}{2}x^2\, dx = \frac{1}{2}e^2 - e.$$

Example 2 Find the volume of the solid bounded by the four planes

$$x = 0 \qquad y = 0 \qquad z = x + y \qquad z = 1 - x - y$$

Step 1 Sketch the planes. We see from Figure 12.3.7 that $z = x + y$ is below $z = 1 - x - y$.

Step 2 Find inequalities for the region .Since the two planes

$$z = x + y \qquad z = 1 - x - y$$

meet at the line $2x + 2y = 1$ $\quad y = \frac{1}{2} - x$

D is the region $0 \le x \le \frac{1}{2}$ $\quad 0 \le y \le \frac{1}{2} - x$

Step 3
$$V = \iint_D (1 - x - y) - (x + y)\, dA = \iint_D 1 - 2x - 2y\, dA$$

$$= \int_0^{1/2} \int_0^{1/2 - x} 1 - 2x - 2y\, dy\, dx$$

$$\int_0^{1/2-x} 1 - 2x - 2y\, dy = y - 2xy - y^2 \Big]_0^{1/2-x}$$

$$= \frac{1}{2} - x - 2x(\frac{1}{2} - x)^2 = \frac{1}{4} - x + x^2$$

$$V=\int_0^{1/2}\frac{1}{4}-x+x^2\,dx=\frac{1}{24}$$

Example 3 Find the volume of the solid bounded by the plane $z=2y$ and the paraboloid $z=1-2x^2-y^2$.

Step 1 The surfaces and the region D are sketched in Figure 12.3.8

Step 2 The two surfaces intersect on the curve

$$2y=1-2x^2-y^2$$

or solving for y, $\quad y=-1\pm\sqrt{2-2x^2}$.

Therefore D is the region

$$-1\le x\le 1,\qquad -1-\sqrt{2-2x^2}\le y\le -1+\sqrt{2-2x^2}.$$

Step 3 We see from the figure that the plane is lower surface and the paraboloid is the upper surface.

$$V=\iint_D (1-2x^2-y^2)-2y\,dA$$

$$=\int_{-1}^{1}\int_{-1-\sqrt{2-2x^2}}^{-1+\sqrt{2-2x^2}}(1-2x^2-y^2-2y)\,dy\,dx.$$

$$\int_{-1-\sqrt{2-2x^2}}^{-1+\sqrt{2-2x^2}}(1-2x^2-y^2-2y)\,dy=\int_{-\sqrt{2-2x^2}}^{\sqrt{2-2x^2}}(2-2x-u^2)\,du$$

$$=\frac{8\sqrt{2}}{3}(1-x^2)^{3/2}.$$

$$V=\int_{-1}^{1}\frac{8\sqrt{2}}{3}(1-x^2)^{3/2}\,dx.$$

Put $x=\sin\theta$, $\sqrt{1-x^2}=\cos\theta$, $dx=\cos\theta\,d\theta$

$$V=\frac{8\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$=\frac{8\sqrt{2}}{3}\left(\frac{1}{4}\cos^3\theta\sin\theta+\frac{3}{4}\left(\frac{1}{2}\cos\theta\sin\theta+\frac{1}{2}\theta\right)\right)\Big]_{-\pi/2}^{\pi/2}$$

$$=\frac{8\sqrt{2}}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi=\sqrt{2}\pi$$

Answer $V=\sqrt{2}\pi$

12.4 APPLICATIONS TO PHYSICS

Example 1 Find the mass of an object in the shape of a unit square whose density is the sum of the distance from one edge and twice the distance from a second perpendicular edge.

Step 1 The region D is shown in Figure 12.4.2

Step 2 Place the object so the first two edges are on the x and y axes. Then D is the region

$$0 \le x \le 1, \qquad 0 \le y \le 1.$$

Step 3 The density is $\rho(x,y) = y + 2x$,

$$m = \iint_D y + 2x\, dA = \int_0^1 \int_0^1 y + 2x\, dy\, dx.$$

$$\int_0^1 y + 2x\, dy = \frac{1}{2}y^2 + 2xy \Big]_0^1 = \frac{1}{2} + 2x$$

$$m = \int_0^1 \frac{1}{2} + 2x\, dx = \frac{3}{2}$$

Example 2 A triangular plate bounded by the lines $x = 0$, $x = y$, $y = 1$ has density $\rho(x,y) = x + y$ .Find the moments and center of mass.

Step 1 Sketch the region D, as in Figure 12.4.4.

Step 2 We see from the figure that D is the region

$$0 \le x \le 1, \qquad x \le y \le 1.$$

Step 3 Set up and evaluate the iterated integrals for the mass m and moments M_x and M_y.

$$m = \iint_D x + y\, dA = \int_0^1 \int_x^1 x + y\, dy\, dx$$

$$\int_x^1 x + y\, dy = x + \frac{1}{2} - \frac{3}{2}x^2.$$

$$m = \int_0^1 x + \frac{1}{2} - \frac{3}{2}x^2 dx = \frac{1}{2}.$$

$$M_x = \iint_D y(x+y)\, dA = \int_0^1 \int_x^1 yx + y^2\, dy\, dx$$

$$\int_x^1 yx + y^2\, dy = \frac{1}{2}x + \frac{1}{3} - \frac{5}{6}x^3.$$

$$M_x = \int_0^1 \frac{1}{2}x + \frac{1}{3} - \frac{5}{6}x^3\,dx = \frac{9}{24}.$$

$$M_y = \iint_D x(x+y)\,dA = \int_0^1\int_x^1 x^2 + xy\,dy\,dx$$

$$\int_x^1 x^2 + xy\,dy = x^2 + \frac{1}{2}x - \frac{3}{2}x^3.$$

$$M_Y = \int_0^1 x^2 + \frac{1}{2}x - \frac{3}{2}x^3\,dx = \frac{5}{24}$$

The answers are $M_x = \frac{9}{24}$, $M_y = \frac{5}{24}$

$$\bar{x} = \frac{M_y}{m} = \frac{5/24}{1/2} = \frac{5}{12.}$$

$$\bar{y} = \frac{M_x}{m} = \frac{9/24}{1/2} = \frac{9}{12.}$$

The point $(\bar{x}, \bar{y})$ is shown in Figure 12.4.5

Example 3 The object in Example 2 is lying horizontally on the ground. Find the work required to stand the object up with the hypotenuse of the triangle on the ground.

We use the formula $W = mgs.$

From Example 2, $m = \frac{1}{2}$ We must find s.

s =minimum distance from $(\frac{5}{12}, \frac{9}{12})$ to the line $x = y$

s =minimum value of $z = \sqrt{(x - \frac{5}{12})^2 + (x - \frac{9}{12})^2}.$

$$z = \sqrt{2x^2 - \frac{28}{12}x + \frac{106}{144}}.$$

$$\frac{dz}{dx} = (4x - \frac{28}{12})\frac{1}{2}z^{-1/2}$$

$$\frac{dz}{dx} = 0 \quad \text{at} \quad 4x = \frac{28}{12}, \qquad x = \frac{7}{12}$$

$$s = \sqrt{2(\frac{7}{12})^2 - \frac{28}{12}\cdot\frac{7}{12} + \frac{106}{144}} = \frac{\sqrt{2}}{6}.$$

$$W = mgs = \frac{1}{2} \cdot g \cdot \frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{12} g.$$

Example 4 Find the moment of inertia about the origin of an object with constant density $\rho = 1$ which covers the square shown in Figure 12.4.7:

$$-\frac{1}{2} \le x \le \frac{1}{2}, \qquad -\frac{1}{2} \le y \le \frac{1}{2}$$

$$I = \iint_D (x^2 + y^2)\, dA = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} x^2 + y^2 \, dy\, dx.$$

$$\int_{-1/2}^{1/2} x^2 + y^2 \, dy = x^2 y + \frac{1}{3} y^3 \Big]_{-1/2}^{1/2} = x^2 + \frac{1}{12}.$$

$$I = \int_{-1/2}^{1/2} x^2 + \frac{1}{12} dx = \frac{1}{6}.$$

12.5 DOUBLE INTEGRALS IN POLAR COORDINATES

Example 1 Find the volume over the unit circle between the surfaces

Step 1 Sketch and the solid, as in Figure 12.5.7.

Step 2 is the polar region

Step 3

For comparison let us also work this problem in rectangular coordinates.

We can see that it is easier to use polar coordinates.

Is the region

We make the trigonometric substitution shown in Figure 12.5.8:

Then

Example 2 Find the mass and center of mass of a flat plate in the shape of a semicircle of radius one whose density is equal to the distance from the center of the circle.

Step 1 The region is sketch in Figure 12.5.9.

Step 2 Take the origin at the center of the circle and the as the base of the semicircle. Is the polar region.

Step 3 The density is

Example 3 Find the moment of inertia of a circle of radius and constant density about the center of the circle.

Step 1 Draw the region

Step 2 Put the origin at the center, so is the polar region

Step 3

12.6 TRIPLE INTEGRALS

Example 1 Evaluate where is the rectangular box

There are six iterated integrals which all have same value. We compute one of them, and then another to check our answer.

First solution

The inside integral is

The second integral is

The final answer is

Second solution

The inside integral is

The second integral is

The final answer is

Triple integrals can be evaluated by iterated integrals.

Example 2 Evaluate where is the region shown in Figure 12.6.5,

Solution

We first evaluate the inside integral.

Now we evaluate the second integral.

Finally we evaluate the outside integral.

Example 3 Find the mass of an object in the unit cube

With density

An object in space has a moment about each coordinate plane.

Example 4 An object has constant density and the shape of a tetrahedron with vertices at the four points

Find the center of mass.

Step 1 The region is sketched in Figure 12.6.10.

Step 2 The region is the solid bounded by the coordinate planes and the plane which passes through .Solving for , the planes is

The plane meets the plane at the line , or

Therefore is the region

Step 3 Let the density be

Similarly

The center of mass is

Example 5 Find the moments of inertia about the three axes of an object with constant density 1 filling the cube shown in Figure 12.6.11.

Similarly,

12.7 CYLINDRICAL AND SPHERICAL COORDINATES

Example 1 Find the moment of inertia of a cylinder of height , base a circle of radius , and constant density 1, about its axis.

Step 1 Draw the region as in Figure 12.7.9.

Step 2 The problem is greatly simplified by a wise choice of coordinate axes. Let the be the axis of the cylinder and put the origin at center of the base. Then the region E in rectangular coordinates is

And in cylindrical coordinates is

Step 3 The problem looks easier in cylindrical coordinates.

Example 2 Find the center of mass of a cone of constant density with height and base a circle of radius .

Step 1 The region is sketched in Figure 12.7.10.

Step 2 Put the origin at the center of the base and let the z-axis be the axis of the cone. E is the cylindrical region

Step 3 Let the density be 1.

Since the cone is symmetric about the z-axis, and

The point is shown in Figure 12.7.11.

On the other hand, the volume formula in spherical coordinates is something new which is useful for finding volumes of spherical regions.

Example 3 Find the volume of the region above the cone and inside the sphere

The region, shown in Figure 12.7.18, is given by

Example 4 A sphere of diameter passes through the center of a sphere of radius , and .Find the volume of the region inside the sphere of diameter and outside the sphere of radius .

Step 1 The region is sketched in Figure 12.7.19.

Step 2 We let the z-axis be the line through the two centers and put the origin at the center of the sphere of radius . The two spheres have the spherical equations

They intersect at

Thus E is the region

Step 3

Put .Then

Example 5 Find the mass of a sphere of radius whose density is equal to the distance from the surface. The sphere is shown in Figure 12.7.20.

Put the center at the origin. The sphere is then given by

The density at is

density

The mass is