Maxima 在微積分上之應用

偏微分

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11.1 Surfaces

Example 1.

Sketch the portion of the cylinder $x^2 + y^2 = 1$ where $1 \le z \le 2$ (Figure 11.1.12)

Solution：(%i1) load(draw); 因爲 Maxima 內沒有直接畫圓柱體的指令，因此，要畫出圓柱體需要讀取模組 draw，用模組 draw 內建的指令去畫圖 //讀取模組 draw

```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp
```

(%i2) draw3d(cylindrical(1,z,1,2,az,0,2*%pi)); 畫圓柱體的指令：draw3d(cylindrical(半徑，z 座標，最小 z 值，最大 z 值，azi，最小 azi 值，最大 azi 值))，在此(z，azi)代表圓柱體的座標 //此題半徑爲 1，z 軸的範圍爲 1~2

```
(%o2) [gr3d(cylindrical)]
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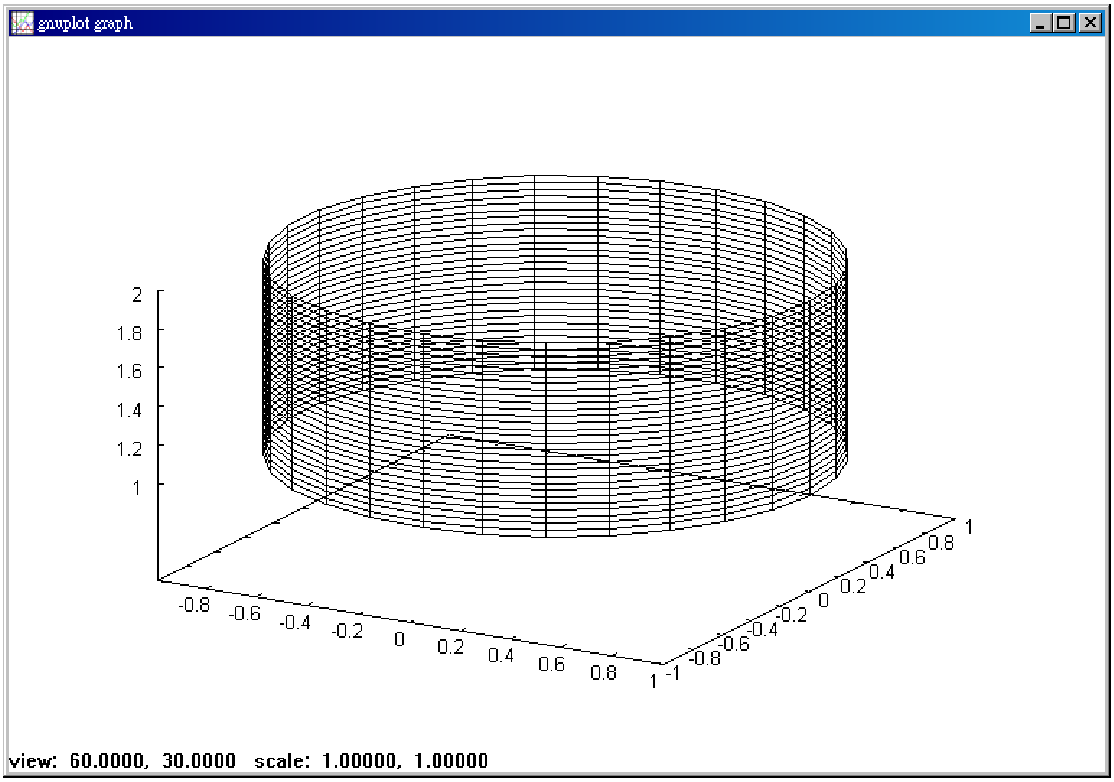


Step 1： Draw the curve $x^2 + y^2 = 1$ in the (x, y) plane. The curve is a circle of radius one.

Step 2： Draw the three coordinate axes and the horizontal planes $z = 1, z = 2$.

Step 3： Draw the circles $x^2 + y^2 = 1$ where the surface intersects the two planes $z = 1, z = 2$.

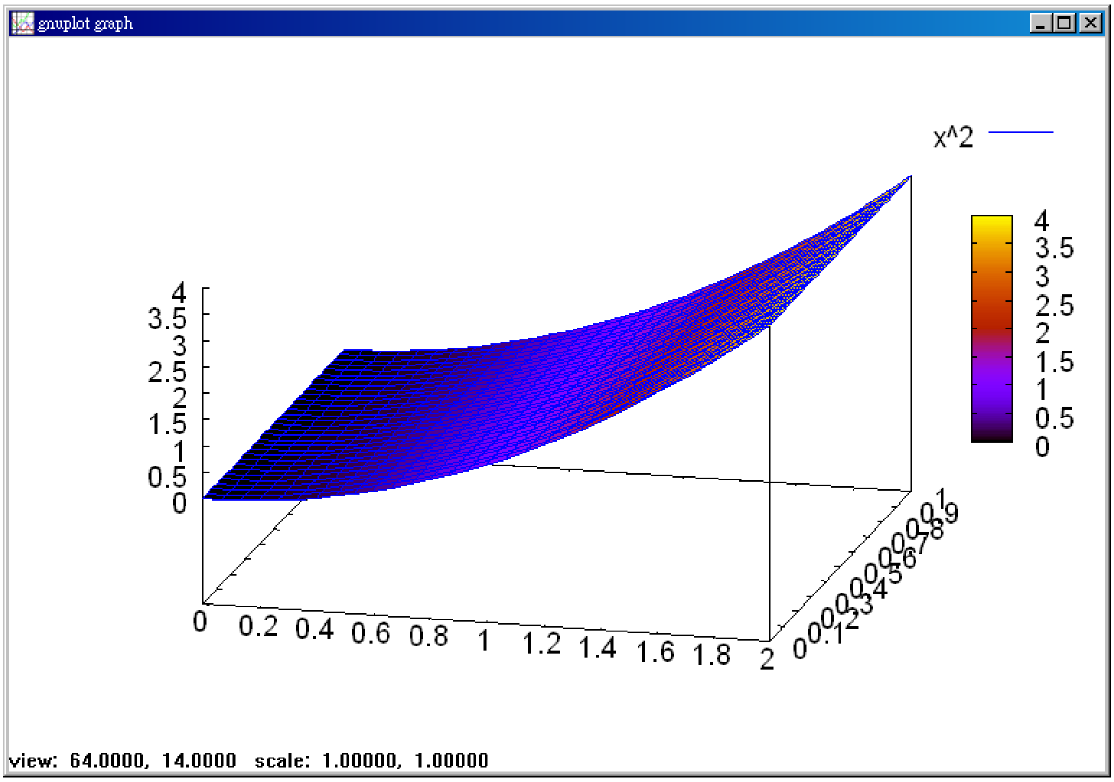
Step 4： Complete the sketch by drawing heavy lines for all edges which would be visible on an "opaque" model of the given surface. This surface is called a circular cylinder.

Example 2.

Sketch the part of the cylinder $z = x^2$ where $0 \le y \le 2, 0 \le z \le 1$. This is a parabolic cylinder parallel to the y-axis, because y does not appear in the equation. The four steps are shown in Figure 11.1.13.

Solution：(%i1) plot3d(x^2,[x,0,2],[z,0,1],[grid,30,30]); 畫 3d 圖形的指令：plot3d(函數式，x 軸的範圍，y 軸的範圍，z 軸的範圍，其他繪圖選項) //函數式爲 $z = x^2$ ，x 軸的範圍爲 1~2，z 軸的範圍爲 0~1，網格劃分爲 30×30

(%o1)



For sketching the graph of a function $z = f(x, y)$, a topographic map, or contour map, can often be used as a first step. It is a method of representing a surface which is often found in atlases. In a topographic map, the curves $f(x, y) = z_0$ are sketched in the (x, y) plane for several different constant z_0, and each curve is labeled (Figure 11.1.14). These curves are called level curves, or contours.

Example 3.

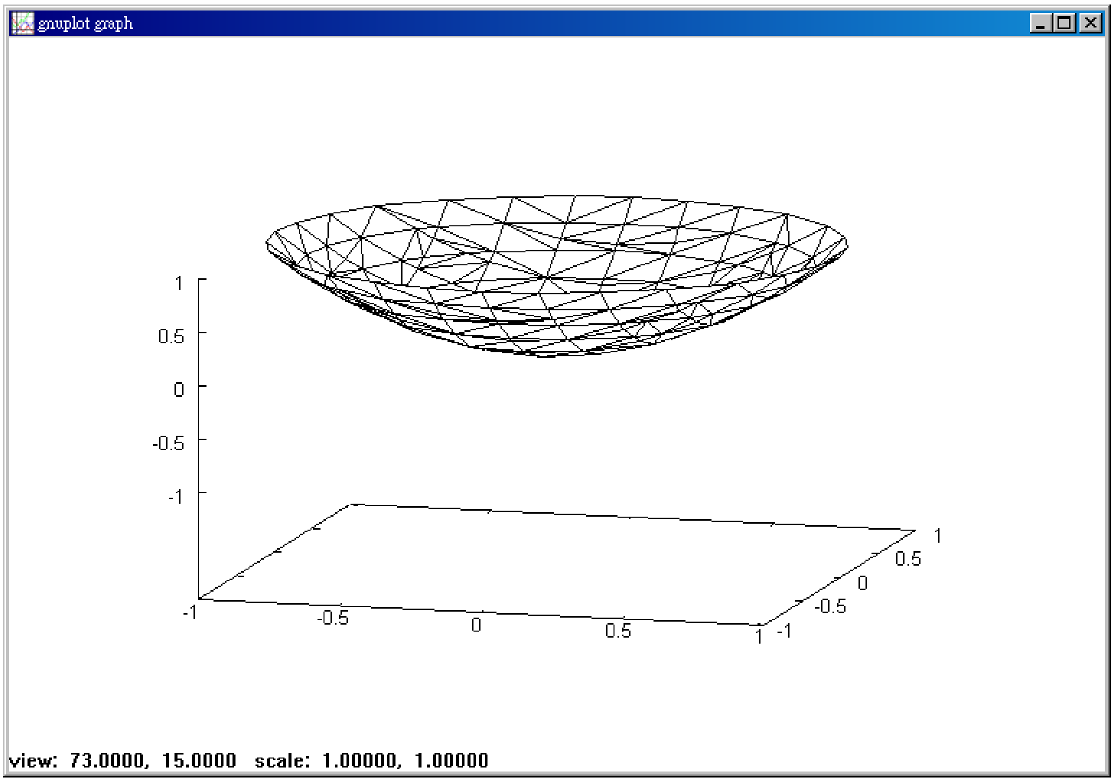
Sketch the part of the surface $z = x^2 + y^2$ where $-1 \le z \le 1$. This is an elliptic paraboloid (Figure 11.1.15).

Solution：(%i1) load(draw); 因爲 Maxima 內沒有直接畫圓柱體的指令，因此，要畫出圓柱體需要讀取模組 draw，用模組 draw 內建的指令去畫圖 //讀取模組 draw

```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp
```

(%i2) draw3d(implicit(x^2+y^2=z,x,-1,1,y,-1,1,z,-1,1),surface_hide=true); 畫 3d 隱函數的曲面圖指令：draw3d(implicit(函數式，x 軸範圍，y 軸範圍，z 軸範圍)，surface_hide=true)，若 surface_hide=true 則隱藏一部分不畫在 3d 曲面圖上 //此題函數式為 $z = x^2 + y^2$，z 軸範圍為-1~1

(%o2) [gr3d(implicit)]



Step 1： Draw the topographic map. The level curves are circles.

Step 2： Draw the axes and the planes $z = -1, z = 1$.

Step 3： Draw the intersections of the surface with the planes $z = -1, z = 1$ and also the planes $x = 0$ and $y = 0$.

$z = -1$： No intersection.

$z = 1$： The circle $x^2 + y^2 = 1$.

$x = 0$： The parabola $z = y^2$.

$y = 0$： The parabola $z = x^2$.

Step 4： Complete the figure with heavy lines for visible edges.

Example 4.

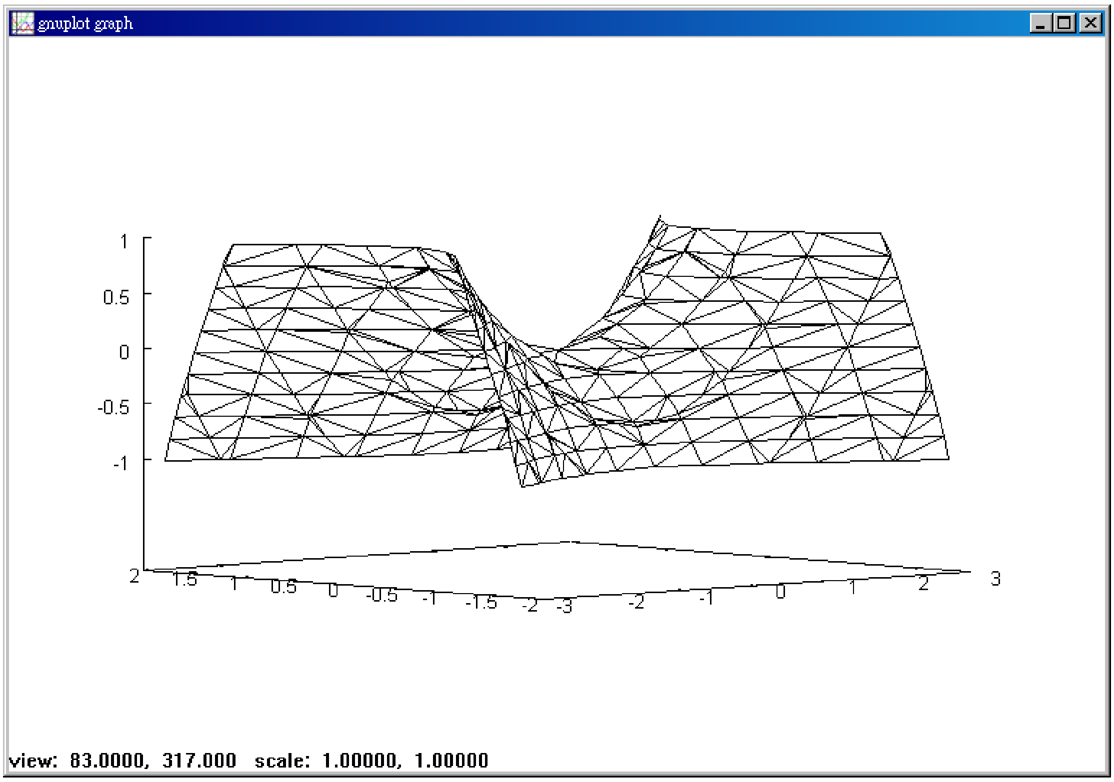
Graph the function $\frac{x^2}{4} - y^2 = z$, where $-3 \le x \le 3$, $-2 \le y \le 2$, $-1 \le z \le 1$. This is a hyperbolic paraboloid (Figure 11.1.16).

Solution：(%i1) load(draw); 因為 Maxima 內沒有直接畫圓柱體的指令，因此，要畫出圓柱體需要讀取模組 draw，用模組 draw 內建的指令去畫圖 //讀取模組 draw

(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp

(%i2) draw3d(implicit((x^2)/4-y^2=z,x,-3,3,y,-2,2,z,-1,1),surface_hide=true); 畫 3d 隱函數的曲面圖指令：draw3d(implicit(函數式，x 軸範圍，y 軸範圍，z 軸範圍)，surface_hide=true)，若 surface_hide=true 則隱藏一部分不畫在 3d 曲面圖上 //此題函數式為 $\frac{x^2}{4} - y^2 = z$，x 軸範圍為-3~3，y 軸範圍為-2~2，z 軸範圍為-1~1

(%o2) [gr3d(implicit)]



Step 1： Draw a topographic map. The level curves are hyperbolas.

Step 2： Draw the axes and rectangular solid.

Step 3： Draw the curves where the surface intersects the faces and also the planes $x=0$, $y=0$. The topographic map gives the curves on $z=-1, z=0$, and $z=1$. The curves on $x=0$ and y=0 are parabolas.

Step 4： Complete Figure 11.1.16.

Example 5.

Sketch the surface $-x^2-\frac{y^2}{4}+z^2=1$

Where $-2 \le z \le 2$. Thus is a hyperboloid of two sheets (Figure 11.1.17).

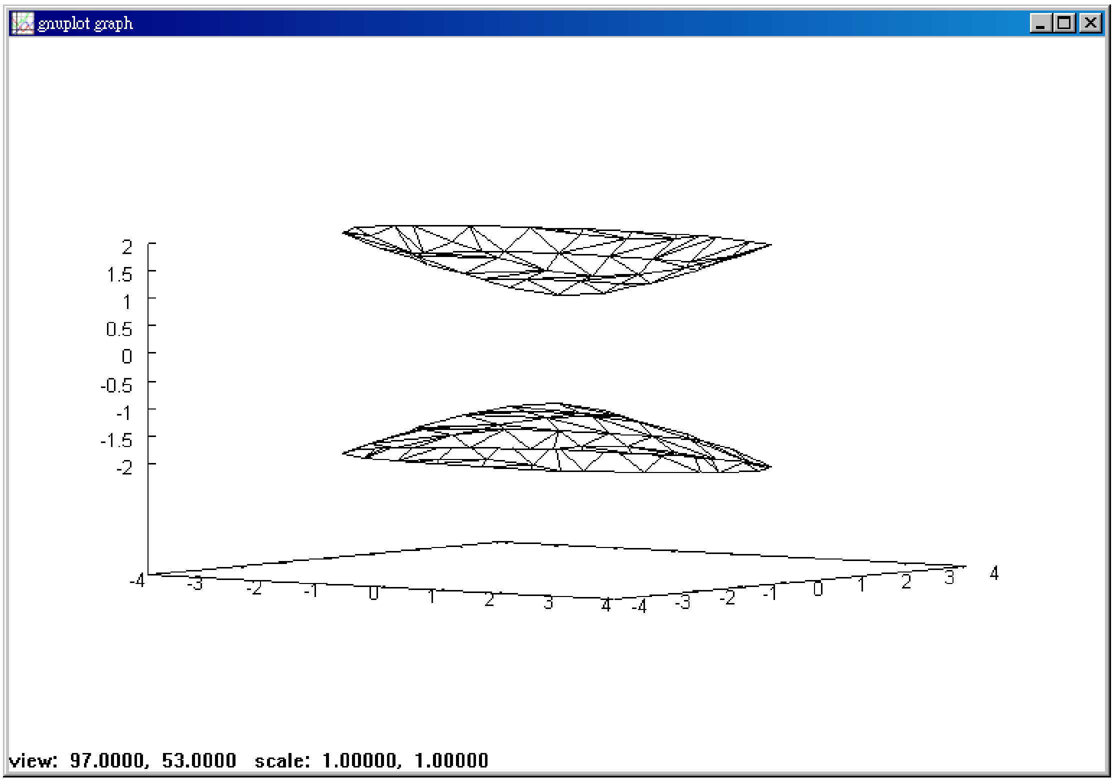
Solution：(%i1) load(draw); 因為 Maxima 內沒有直接畫圓柱體的指令，因此，要畫出圓柱體需要讀取模組 draw，用模組 draw 內建的指令去畫圖 //讀取模組 draw

```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp
```

(%i2) draw3d(implicit(-x^2-(y^2)/4+z^2=1,x,-4,4,y,-4,4,z,-2,2),surface_hide=true);

畫 3d 隱函數的曲面圖指令：draw3d(implicit(函數式，x 軸範圍，y 軸範圍，z 軸範圍)，surface_hide=true)，若 surface_hide=true 則隱藏一部分不畫在 3d 曲面圖上 //此題函數式為 $-x^2-\frac{y^2}{4}+z^2=1$，z 軸範圍為-2~2

```
(%o2) [gr3d(implicit)]
```



Although it is not a function, it can be broken up into two functions

$$z=\sqrt{1+x^2+\frac{y^2}{4}},\quad z=-\sqrt{1+x^2+\frac{y^2}{4}}.$$

Step 1： Draw topographic maps for $z=\sqrt{1+x^2+y^2/4}$ and $z=-\sqrt{1+x^2+y^2/4}$.

The level curves are ellipses.

Step 2： Draw the axes and the planes $z=2, z=-2$.

Step 3： Draw the intersections of the surface with the planes

$$z=2,\quad z=-2,\quad x=0,\quad y=0.$$

The surface intersects $x=0$ and $y=0$ in the hyperbolas

$$-\frac{1}{4}y^2+z^2=1,\quad -x^2+z^2=1.$$

Step 4： Complete Figure 11.1.17.

Example 6.

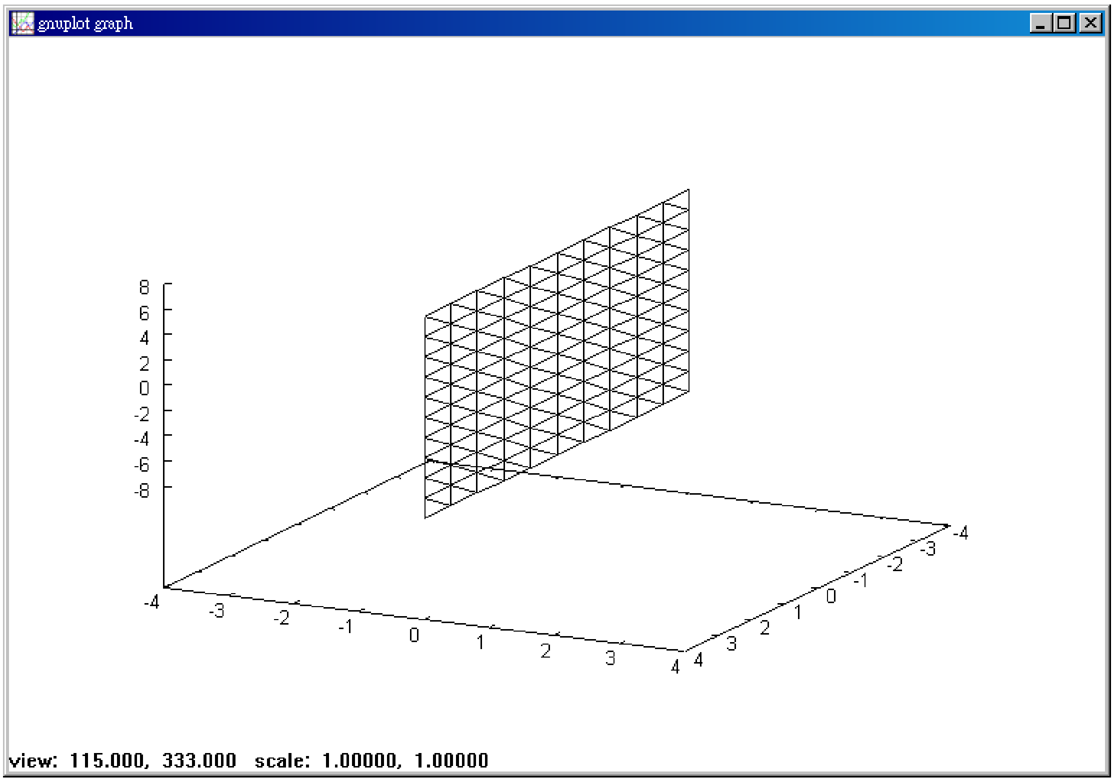
Graph the sum function $z=x+y$. The graph is a plane. A topographic map and sketch of the surface are shown in Figure 11.1.18.

Solution：(%i1) load(draw); 因爲 Maxima 內沒有直接畫圓柱體的指令，因此，要畫出圓柱體需要讀取模組 draw，用模組 draw 內建的指令去畫圖 //讀取模組 draw

```
(%o1)
C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp
```

(%i2) draw3d(implicit(y=x+y,x,-4,4,y,-4,4,y,-8,8),surface_hide=true); 畫 3d 隱函數的曲面圖指令：draw3d(implicit(函數式，x 軸範圍，y 軸範圍，z 軸範圍)，surface_hide=true)，若 surface_hide=true 則隱藏一部分不畫在 3d 曲面圖上 //此題函數式爲 $z = x + y$

(%o2) [gr3d(*implicit*)]



Example 7.

Sketch the graph of the product function $z = xy$, where

$-2 \le x \le 2, \quad -2 \le y \le 2, \quad -1 \le z \le 1.$

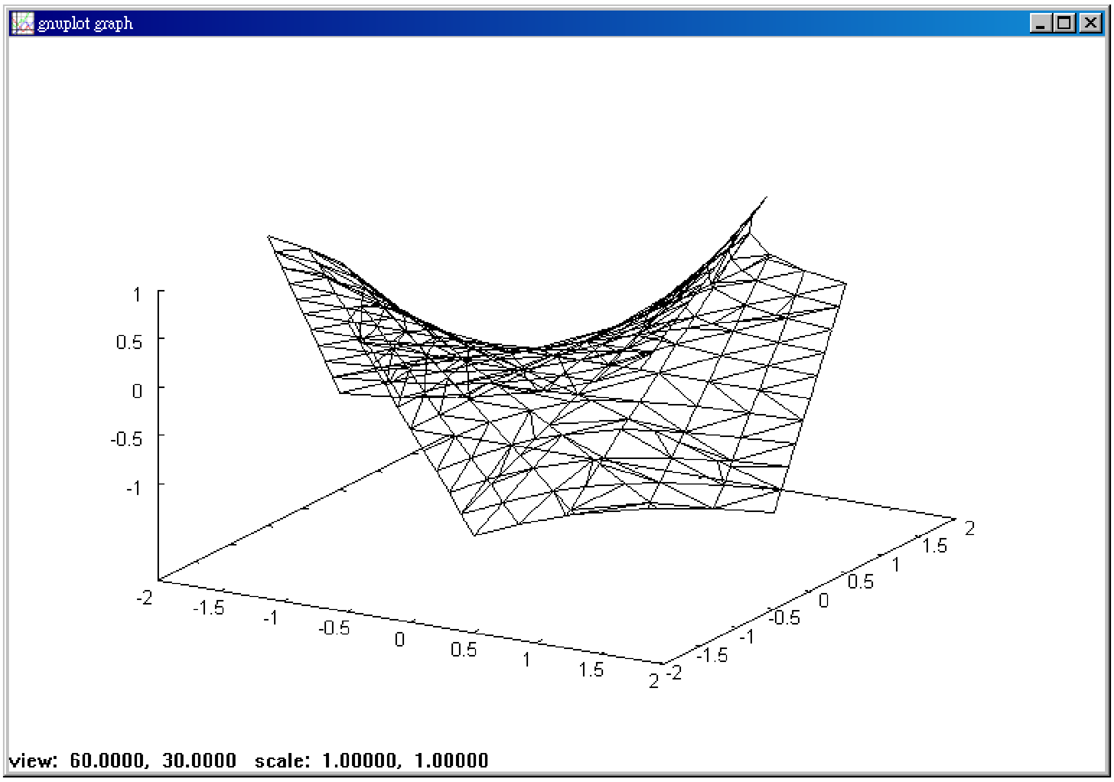
Solution：(%i1) load(draw); 因爲 Maxima 內沒有直接畫圓柱體的指令，因此，要畫出圓柱體需要讀取模組 draw，用模組 draw 內建的指令去畫圖 //讀取模組

draw

(%o1) C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp

(%i2) draw3d(implicit(z=x*y,x,-2,2,y,-2,2,z,-1,1)); 畫 3d 隱函數的曲面圖指令：draw3d(implicit(函數式，x 軸範圍，y 軸範圍，z 軸範圍)，surface_hide=true)，若 surface_hide=true 則隱藏一部分不畫在 3d 曲面圖上 //此題函數式爲 $z = xy$ ，x 軸範圍爲-2~2，y 軸範圍爲-2~2，z 軸範圍爲-1~1

(%o2) [gr3d(implicit)]



The surface is saddle shaped. It intersects the horizontal plane $z = z_0$ in the curve $y = z_0 / x$. It intersects the vertical planes $x = x_0$ and $y = y_0$ in the lines $z = x_0 y$ and $z = xy_0$. The surface is shown in Figure 11.1.19.

Example 8.

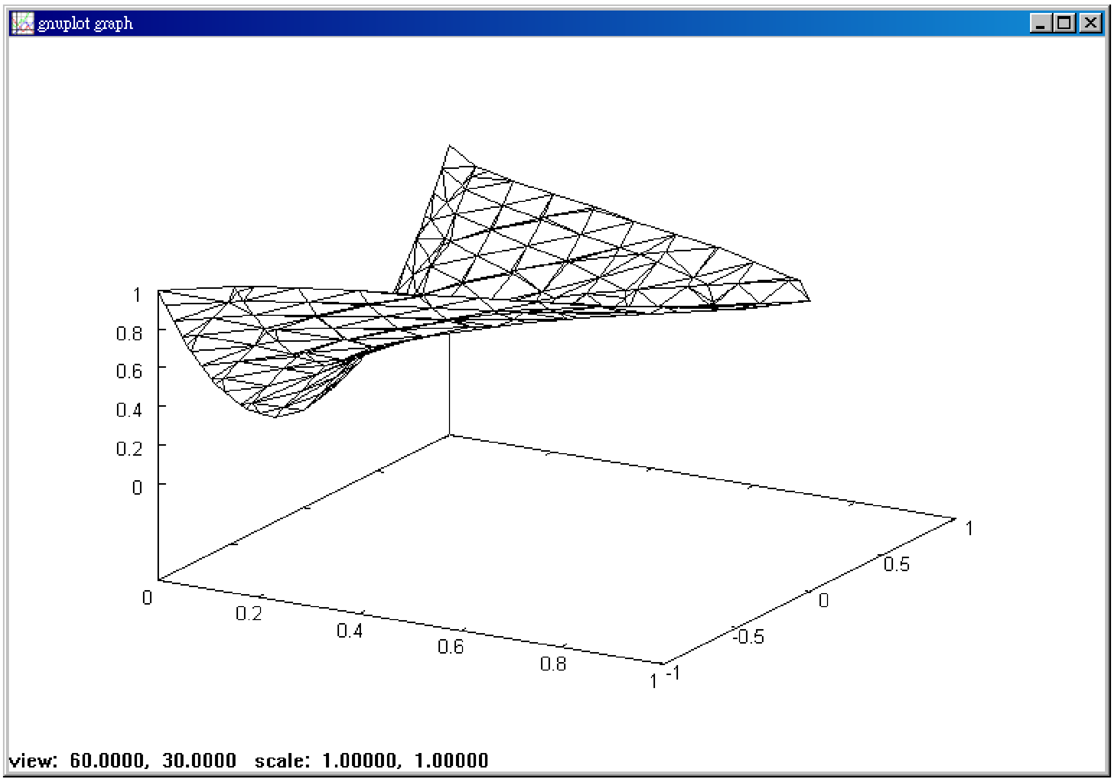
Graph the function $z=\sqrt{x}+y^2$ where $0\le x\le 1,\quad -1\le y\le 1,\quad 0\le z\le 1$.

Solution：(%i1) load(draw); 因為 Maxima 內沒有直接畫圓柱體的指令，因此，要畫出圓柱體需要讀取模組 draw，用模組 draw 內建的指令去畫圖 //讀取模組 draw

(%o1) C:/PROGRA~1/MAXIMA~1.2/share/maxima/5.19.2/share/draw/draw.lisp

(%i2) draw3d(implicit(z=sqrt(x)+y^2,x,0,1,y,-1,1,z,0,1),surface_hide = true); 畫 3d 隱函數的曲面圖指令：draw3d(implicit(函數式，x 軸範圍，y 軸範圍，z 軸範圍)，surface_hide=true)，若 surface_hide=true 則隱藏一部分不畫在 3d 曲面圖上 //此題函數式為 $z=\sqrt{x}+y^2$ ，x 軸範圍為 0~1，y 軸範圍為-1~1，z 軸範圍為 0~1

(%o2) [gr3d(implicit)]



Step 1： The topographic map has level curves

$$\sqrt{x}+y^2=c,\quad x=(c-y^2)^2 \text{ with } y^2\le c.$$

The derivative $dx/dy=4y(c-y^2)$ has zeros at $y=0$ and $y=\pm\sqrt{c}$.

The table shows that the curves are bell shaped.

y	x	dx/dy	
$-\sqrt{c}$	0	0	Min
0	c^2	0	Max
$\sqrt{c}$	0	0	Min

Step 2： Draw the rectangular solid.

Step 3： The surface intersects the plane $x = 0$ in the parabola $z = y^2$, and intersects the plane $y = 0$ in the curve $z = \sqrt{x}$. It intersects the plane $z = 1$ in the curve $x = (1 - y^2)^2$.

Step 4： The surface, shown in Figure 11.1.20, is shaped like a breaker spout.

<u>11.2 Continuous Functions of Two or More Variables</u>

Example 1.

Show that $f(x, y) = 2x + xy^2$ is continuous for all (a, b). Let $st(x) = a$ and $st(y) = b$. Then

$$st(2x + xy^2) = st(2x) + st(xy^2) = 2st(x) + st(x)st(y^2) = 2a + ab^2.$$

Here is a list of important continuous functions of two variables.

Example 2.

By (I), $h(x, y) = \sin(x + y)$ is continuous for all (x, y).

Example 3.

By (II), $h(x, y) = \sin x \cos y$ is continuous for all (x, y).

Example 4.

Find a set on which $h(x,y)=\ln(x+y)$ is continuous.

By Theorem 1 and 2, $x+y$ is continuous for all (x,y),

$\ln u$ is continuous for $u>0$,

$\ln(x+y)$ is continuous for $x+y>0$.

Answer $\ln(x+y)$ is continuous on the set of all (x,y) such that $x+y>0$, sown in Figure 11.2.3.

Example 5.

Find a set on which $h(x,y)=x^y+\cos\sqrt{x^2-y}$ is continuous.

x^y is continuous for $x>0$.

x^2 is continuous for all x.

x^2-y is continuous for all (x,y).

$\sqrt{x^2-y}$ is continuous for $x^2-y>0$.

$\cos\sqrt{x^2-y}$ is continuous for $x^2-y>0$.

$x^y + \cos\sqrt{x^2 - y}$ is continuous for $x > 0$ and $x^2 - y > 0$.

Answer $h(x, y)$ is continuous on the set of all (x, y) such that $x > 0$ and $x^2 - y > 0$. The set is shown in Figure 11.2.4.

Example 6.

Find a set on which $h(x, y) = \log_x y$ is continuous.

We use the identity $\log_x y = \dfrac{\ln y}{\ln x}$.

$\ln y$ is continuous for $y > 0$.

$\ln x$ is continuous for $x > 0$.

$\ln y / \ln x$ is continuous for $x > 0, \quad \ln x \neq 0, \quad y > 0,$

that is, $x > 0, \quad x \neq 1, \quad y > 0$.

$\log_x y$ is continuous for $x > 0, \quad x \neq 1, \quad y > 0$.

Answer $\log_x y$ is continuous on the set of all (x, y) such that $x > 0, \quad x \neq 1, \quad y > 0$ (Figure 11.2.5).

Example 7.

Find a set where the function $h(x, y, z) = \dfrac{x^2 y}{x + y + z}$ is continuous.

x^2 is continuous for all x.

$x^2 y$ is continuous for all (x, y).

$x + y$ is continuous for all (x, y).

$(x + y) + z$ is continuous for all (x, y, z).

$\dfrac{x^2 y}{x + y + z}$ is continuous for all $x + y + z \neq 0$.

Answer $h(x, y, z)$ is continuous on the set of all (x, y, x) such that $x + y + z \neq 0$.

11.3 Partial Derivatives

Example 1.

Find the partial derivatives of the function

$f(x, y) = x^2 + 3xy - 8y$

at the point $(2,-1)$.

Solution：(%i1) f(x,y):=x^2+3*x*y-8*y; //建立一函數 $x^2 + 3xy - 8y$，方程式名稱叫做 f(x,y)

(%o1) $f(x, y) := x^2 + 3\,x\,y + (-8)\,y$

(%i2) diff(f(x,y),x); 微分的指令：differ(函數，要微分的變數) //對函數 f(x,y)中的 x 變數微分

(%o2) 3 y+2 x

(%i3) g1(x,y):=3*y+2*x; //將對 x 偏微分後的函數名稱給定為 g1(x,y)

(%o3) g1(x, y):=3 y+2 x

(%i4) g1(2,-1); //將 x=2，y=-1 代入 g1(x,y)求得值 1

(%o4) 1

(%i5) diff(f(x,y),y); 微分的指令：differ(函數，要微分的變數) //對函數 f(x,y) 中的 y 變數微分

(%o5) 3 x-8

(%i6) g2(x,y):=3*x-8; //將對 y 偏微分後的函數名稱給定為 g2(x,y)

(%o6) g2(x, y):=3 x-8

(%i7) g2(2,-1); //將 x=2，y=-1 代入 g2(x,y)求得值-2

(%o7) -2

To find $f_x(x,y)$, we treat y as a constant,

$f_x(x,y) = 2x + 3y$.

To find $f_y(x,y)$, we treat x as a constant,

$f_y(x,y) = 3x - 8$.

Thus $f_x(2,-1) = 2\cdot 2 + 3(-1) = 1$, $f_y(2,-1) = 3\cdot 2 - 8 = -2$.

Figure 11.3.2 shows the surface $z = f(x, y)$ and the tangent lines at the point $(2,-1)$.

Example 2.

A point $P(x, y)$ has distance $z = \sqrt{x^2 + y^2}$ from the origin (Figure 11.3.3). Find the rate of change of z at $P(3,4)$ if：

(a) P moves at unit speed in the x direction.

(b) P moves at unit speed in the y direction.

Solution：(%i1) P(x,y):=sqrt(x^2+y^2); //建立一函數 $\sqrt{x^2 + y^2}$ ，方程式名稱叫做 P(x,y)

(%o1) $P(x, y) := \sqrt{x^2 + y^2}$

(%i2) diff(P(x,y),x); 微分的指令：differ(函數，要微分的變數) //對函數 P(x,y) 中的 x 變數微分

(%o2) $\dfrac{x}{\sqrt{y^2 + x^2}}$

(%i3) g1(x,y):=x/sqrt(y^2+x^2); //將對 x 偏微分後的函數名稱給定爲 g1(x,y)

(%o3) $g1(x, y) := \dfrac{x}{\sqrt{y^2 + x^2}}$

(%i4) g1(3,4);　//將 x=3，y=4 代入 g1(x,y)求得值$\frac{3}{5}$

(%o4) $\frac{3}{5}$

(%i5) diff(P(x,y),y);　微分的指令：differ(函數，要微分的變數)　//對函數 P(x,y)中的 y 變數微分

(%o5) $\frac{y}{\sqrt{y^2+x^2}}$

(%i6) g2(x,y):=y/sqrt(y^2+x^2);　//將對 y 偏微分後的函數名稱給定爲 g2(x,y)

(%o6) g2(x, y):=$\frac{y}{\sqrt{y^2+x^2}}$

(%i7) g2(3,4);　//將 x=3，y=4 代入 g2(x,y)求得值$\frac{4}{5}$

(%o7) $\frac{4}{5}$

In this problem the round d notation is convenient,

(a) $\frac{\partial z}{\partial x}(x,y)=\frac{x}{\sqrt{x^2+y^2}}$,

$$\frac{\partial z}{\partial x}(3,4)=\frac{3}{\sqrt{3^2+4^2}}=\frac{3}{5}.$$

(b) $\frac{\partial z}{\partial y}(x,y)=\frac{y}{\sqrt{x^2+y^2}}$,

$$\frac{\partial z}{\partial y}(3,4)=\frac{4}{\sqrt{3^2+4^2}}=\frac{4}{5}.$$

Example 3.

Find the partial derivative of $f(x,y,z)=\sin(x^2y-z)$ at the point $(1,0,0)$.

The find $f_x(x,y,z)$ we treat y and z as constants.

$f_x(x,y,z)=2xy\cos(x^2y-z)$.

$f_y(x,y,z)=x^2\cos(x^2y-z)$.

$f_z(x,y,z)=-\cos(x^2y-z)$.

Thus $f_x(1,0,0)=2\cdot1\cdot0\cos(1^2\cdot0-0)=0$.

$f_y(1,0,0)=1^2\cdot\cos(1^2\cdot0-0)=1$.

$f_z(0,0,1)=-\cos(1^2\cdot0-0)=-1$.

Solution：(%i1) f(x,y,z):=sin(x^2*y-z); //建立一函數 $\sin(x^2y-z)$，方程式名稱叫做 f(x, y, z)

(%o1) $f(x,y,z):=\sin(x^2\,y-z)$

(%i2) diff(f(x,y,z),x); 微分的指令：differ(函數，要微分的變數) //對函數 f(x, y, z)中的 x 變數微分

(%o2) $2\,x\,y\cos(z-x^2\,y)$

(%i3) g1(x,y,z):=2*x*y*cos(z-x^2*y);　//將對 x 偏微分後的函數名稱給定為 g1(x, y, z)

(%o3) $g1(x,y,z):=2\,x\,y\,\cos(z-x^2\,y)$

(%i4) g1(1,0,0);　//將 x=1，y=0, z=0 代入 g1(x, y, z)求得值 0

(%o4) 0

(%i5) diff(f(x,y,z),y);　微分的指令：differ(函數，要微分的變數)　//對函數 f(x, y, z)中的 y 變數微分

(%o5) $x^2\cos(z-x^2\,y)$

(%i6) g2(x,y,z):=x^2*cos(z-x^2*y);　//將對 y 偏微分後的函數名稱給定為 g2(x, y, z)

(%o6) $g2(x,y,z):=x^2\cos(z-x^2\,y)$

(%i7) g2(1,0,0);　//將 x=1，y=0, z=0 代入 g2(x, y, z)求得值 1

(%o7) 1

(%i8) diff(f(x,y,z),z);　微分的指令：differ(函數，要微分的變數)　//對函數 f(x, y, z)中的 z 變數微分

(%o8) $-\cos(z-x^2\,y)$

(%i9) g3(x,y,z):=-cos(z-x^2*y);　//將對 z 偏微分後的函數名稱給定為 g3(x, y, z)

(%o9) $g3(x,y,z):=-\cos(z-x^2\,y)$

(%i10) g3(1,0,0);　//將 x=1，y=0, z=0 代入 g3(x, y, z)求得值-1

```
(%o10) -1
```

11.4　Total Differentials and Tangent Planes

Example 1.

Find the increment and total differential of the product function $z = xy$ (Figure 11.4.2).

Solution：(%i1) diff(x*y);　微分的指令：differ(函數，要微分的變數)　//對函數 xy 微分，但此題並沒有特定對某個變數微分，所傳回的值我們稱為"total differential"，del(x)代表對 x 變數微分，del(y)代表對 y 變數微分

```
(%o1) x del(y)+y del(x)
```

Increment： $\Delta z = (x+\Delta x)(y+\Delta y) - xy = y\Delta x + x\Delta y + \Delta x\Delta y$.

Total differential： $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = ydx + xdy$.

Example 2.

Find the increment and total differential of $z = x^2 - 3xy^2$.

Solution：(%i1) diff(x^2-3*x*y^2);

微分的指令：differ(函數，要微分的變數)　//對函數 $x^2 - 3xy^2$ 微分，但此題並沒有特定對某個變數微分，所傳回的值我們稱為"total differential"，del(x)代表對 x

變數微分，del(y)代表對 y 變數微分

(%o1) (2 x−3 y^2) del(x)−6 x y del(y)

Increment： $\Delta z=[(x+\Delta x)^2-3(x+\Delta x)(y+\Delta y)^2]-[x^2-3xy^2]$

$$=[x^2+2x\Delta x+\Delta x^2-3xy^2-6xy\Delta y-3x\Delta y^2-3\Delta xy^2$$

$$-6\Delta xy\Delta y-3\Delta x\Delta y^2]-[x^2-3xy^2]$$

$$=2x\Delta x+\Delta x^2-6xy\Delta y-3x\Delta y^2-3\Delta xy^2-6\Delta xy\Delta y-3\Delta x\Delta y^2$$

$$=(2x-3y^2)\Delta x-6xy\Delta y+\Delta x^2-3x\Delta y^2-6y\Delta x\Delta y-3\Delta x\Delta y^2.$$

Total differential： $\frac{\partial z}{\partial x}=2x-3y^2,\quad \frac{\partial z}{\partial y}=-6xy.$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=(2x-3y^2)dx-6xydy.$$

Example 1. (Continued)

The product function $z=xy$ is smooth for all (x,y).

Express Δz in the form

$\Delta z=dz+\varepsilon_1\Delta x+\varepsilon_2\Delta y$.

We have $\Delta z=y\Delta x+x\Delta y+\Delta x\Delta y$,

$dz=y\Delta x+x\Delta y$.

Thus $\Delta z=dz+\Delta x\cdot\Delta y$.

The problem has more than one correct answer. One answer is $\varepsilon_1 = 0$ and $\varepsilon_2 = \Delta x$, so that

$$\Delta z = dz + 0 \cdot \Delta x + \Delta x \cdot \Delta y = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y .$$

Another answer is $\varepsilon_1 = \Delta y$ and $\varepsilon_2 = 0$, so that

$$\Delta z = dz + \Delta y \cdot \Delta x + 0 \cdot \Delta y = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y .$$

Example 2. (Continued)

The function $z = x^2 - 3xy^3$ is smooth for all (x, y).

Express Δz in the form

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

at an arbitrary point (x, y) and at the point (5, 4). We have

$$\Delta z = (2x - 3y^2)\Delta x - 6xy\Delta y + \Delta x^2 - 3x\Delta y^2 - 6y\Delta x\Delta y - 3\Delta x\Delta y^2 ,$$

$$dz = (2x - 3y^2)\Delta x - 6xy\Delta y .$$

Then $\Delta z = dz + \Delta x^2 - 3x\Delta y^2 - 6y\Delta x\Delta y - 3\Delta x\Delta y^2$.

Each term after the dz has either a Δx or a Δy or both. Factor Δx from all the terms where Δx appears and Δy from the remaining terms.

$$\Delta z = dz + (\Delta x - 6y\Delta y - 3\Delta y^2)\Delta x + (-3x\Delta y)\Delta y .$$

Then $\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, where $\varepsilon_1 = \Delta x - 6y\Delta y - 3\Delta y^2$, $\varepsilon_2 = -3x\Delta y$.

At the point (5, 4),

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 = \Delta x - 24\Delta y - 3\Delta y^2$, $\varepsilon_2 = -15\Delta y$.

Example 3.

Find the equation of the tangent plane to

$$z = 1 + \sin(2x + 3y)$$

At the point (0, 0).

Solution：(%i1) z(x,y):=1+sin(2*x+3*y); //建立一函數$1+\sin(2x+3y)$，方程式名稱叫做 z(x, y)

```
(%o1) z(x,y):=1+sin(2 x+3 y)
```

(%i2) diff(z(x,y),x); 微分的指令：differ(函數，要微分的變數) //對函數 z(x, y)中的 x 變數微分

```
(%o2) 2 cos(3 y+2 x)
```

(%i3) g1(x,y):=2*cos(3*y+2*x); //將對 x 偏微分後的函數名稱給定爲 g1(x, y)

```
(%o3) g1(x,y):=2 cos(3 y+2 x)
```

(%i4) g1(0,0); //將 x=0，y=0 代入 g1(x, y)求得值 2

(%o4) 2

(%i5) diff(z(x,y),y); 微分的指令：differ(函數，要微分的變數) //對函數 z(x, y) 中的 y 變數微分

(%o5) 3 cos(3 y+2 x)

(%i6) g2(x,y):=3*cos(3*y+2*x); //將對 y 偏微分後的函數名稱給定爲 g2(x, y)

(%o6) g2(x, y):=3 cos(3 y+2 x)

(%i7) g2(0,0); //將 x=0，y=0 代入 g2(x, y)求得值 3

(%o7) 3

We have $\frac{\partial z}{\partial x}(x, y) = 2\cos(2x+3y)$, $\frac{\partial z}{\partial y}(x, y) = 3\cos(2x+3y)$.

At the point (0, 0), $z = 1+\sin(0+0) = 1$,

$\frac{\partial z}{\partial x}(0,0) = 2\cos(0+0) = 2$, $\frac{\partial z}{\partial y}(0,0) = 3\cos(0+0) = 3$.

The equation of the tangent plane is $z-1 = 2(x-0)+3(y-0)$, or $z = 2x+3y+1$.

Example 4.

Find the tangent plane to the sphere $x^2 + y^2 + z^2 = 14$

at the point (1, 2, 3) (Figure 11.4.7).

Solution：(%i1) z(x,y):=sqrt(14-x^2-y^2);　//建立一函數 $\sqrt{14-x^2-y^2}$ ，方程式名稱叫做 z(x, y)

(%o1) $z(x,y):=\sqrt{14-x^2-y^2}$

(%i2) diff(z(x,y),x);　微分的指令：differ(函數，要微分的變數)　//對函數 z(x, y) 中的 x 變數微分

(%o2) $-\frac{x}{\sqrt{-y^2-x^2+14}}$

(%i3) g1(x,y):=x/sqrt(-y^2-x^2+14);　//將對 x 偏微分後的函數名稱給定爲 g1(x, y)

(%o3) $g1(x,y):=\frac{x}{\sqrt{-y^2-x^2+14}}$

(%i4) g1(1,2);　//將 x=1，y=2 代入 g1(x, y)求得值 $\frac{1}{3}$

(%o4) $\frac{1}{3}$

(%i5) diff(z(x,y),y);　微分的指令：differ(函數，要微分的變數)　//對函數 z(x, y) 中的 y 變數微分

(%o5) $-\frac{y}{\sqrt{-y^2-x^2+14}}$

(%i6) g2(x,y):=y/sqrt(-y^2-x^2+14);　//將對 y 偏微分後的函數名稱給定爲 g2(x, y)

(%o6) $g2(x,y):=\frac{y}{\sqrt{-y^2-x^2+14}}$

(%i7) g2(1,2);　//將 x=1，y=2 代入 g2(x, y)求得值 $\frac{2}{3}$

(%o7) $\frac{2}{3}$

The top hemisphere has the equation $z=\sqrt{14-x^2-y^2}$.

Then $\frac{\partial z}{\partial x}(x,y)=-\frac{x}{\sqrt{14-x^2-y^2}}=-\frac{x}{z}$,

$$\frac{\partial z}{\partial y}(x,y)=-\frac{y}{\sqrt{14-x^2-y^2}}=-\frac{y}{z}.$$

At (1, 2), $z=3$, $\frac{\partial z}{\partial x}(1,2)=-\frac{1}{3}$, $\frac{\partial z}{\partial y}(1,2)=-\frac{2}{3}$.

Then the tangent plane has the equation

$$z-3=\frac{\partial z}{\partial x}(x-1)+\frac{\partial z}{\partial y}(y-2),$$

$$z-3=-\frac{1}{3}(x-1)+(-\frac{2}{3})(y-2),$$

or $x+2y+3z=14$.

Example 5.

Given $w=xyz$, express the increment Δw in the form

$$\Delta w=dw+\varepsilon_1\Delta x+\varepsilon_2\Delta y+\varepsilon_3\Delta z.$$

We first find Δw and dw,

$$\Delta w=(x+\Delta x)(y+\Delta y)(z+\Delta z)-xyz$$

$$= yz\Delta x + xz\Delta y + xy\Delta z + x\Delta y\Delta z + y\Delta x\Delta z + z\Delta x\Delta y + \Delta x\Delta y\Delta z$$

$$\frac{\partial w}{\partial x} = yz, \quad \frac{\partial w}{\partial y} = xz, \quad \frac{\partial w}{\partial z} = xy.$$

Thus $\Delta w = dw + (y\Delta z + z\Delta y)\Delta x + (x\Delta z)\Delta y + (\Delta x\Delta y)\Delta z$.

Figure 11.4.8 pictures dw and Δw.

11.5 Chain Rule

Example 1.

A particle moves in such a way that

$$\frac{dx}{dt} = 6, \quad \frac{dy}{dt} = -2.$$

Find the rate of change of the distance from the particle to the origin when the particle is at the point (3, -4) (Figure 11.5.2).

Solution：(%i1) z:sqrt(x^2+y^2); //建立一變數 $\sqrt{x^2+y^2}$ ，名稱叫做 z

(%o1) $\sqrt{y^2+x^2}$

(%i2) depends ([x,y],t); 相依的指令：depends(函數，相依的變數) //此題 x 和 y 都和 t 有關係

(%o2) [x(t), y(t)]

(%i3) diff(z,t); 微分的指令：differ(函數，要微分的變數) //對變數 z 中的 t 變數微分

(%o3) $\dfrac{2\, y\left(\frac{d}{d\,t}\, y\right) + 2\, x\left(\frac{d}{d\,t}\, x\right)}{2\sqrt{y^2+x^2}}$

$z = \sqrt{x^2+y^2}$,

$$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}},\quad \frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}.$$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\frac{6x}{\sqrt{x^2+y^2}}-\frac{2y}{\sqrt{x^2+y^2}}$$

$$=\frac{6\cdot 3}{\sqrt{3^2+4^2}}-\frac{2\cdot(-4)}{\sqrt{3^2+4^2}}=\frac{26}{5}.$$

Example 2.

Find the derivative of $z=\sqrt[t]{\sin t}$, using the Chain Rule. (This can also be done by logarithmic differentiation.)

Solution：(%i1) z:(sin(t))^(1/t);　　//建立一變數 $\sqrt[t]{\sin t}$ ，名稱叫做 z

(%o1) $\sin(t)^{1/t}$

(%i2) depends ([x,y],t);　相依的指令：depends(函數，相依的變數)　　//此題 x 和 y 都和 t 有關係

(%o2) [x(t), y(t)]

(%i3) diff(z,t);　微分的指令：differ(函數，要微分的變數)　//對變數 z 中的 t 變數微分

(%o3) $\sin(t)^{1/t}\left(\frac{\cos(t)}{t\,\sin(t)}-\frac{\log(\sin(t))}{t^2}\right)$

Let $x=\sin t,\quad y=\frac{1}{t}$.

Then $z=x^y$.

$$\frac{\partial z}{\partial x}=yx^{y-1},\quad \frac{\partial z}{\partial y}=(\ln x)x^y,$$

$$\frac{dx}{dt}=\cos t,\quad \frac{dy}{dt}=-\frac{1}{t^2}.$$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= yx^{y-1}\cos t + \ln(x)x^{y}(-\frac{1}{t^2})$$

$$= \frac{\sqrt[t]{\sin t}\cos t}{t\sin t} - \frac{\ln(\sin t)\sqrt[t]{\sin t}}{t^2}.$$

Example 3.

Suppose the price z of steel is proportional to the population x divided by the supply y,

$$z = \frac{cx}{y}.$$

x and y depend on time in such a way that

$$\frac{dx}{dt} = 0.01x, \quad \frac{dy}{dt} = -\sqrt{x}.$$

Solution：(%i1) z:c*x/y; /建立一變數 $\frac{cx}{y}$ ，名稱叫做 z

$$(\%o1) \quad \frac{c\,x}{y}$$

(%i2) depends ([x,y],t); 相依的指令：depends(函數，相依的變數) //此題 x 和 y 都和 t 有關係

(%o2) [x(t), y(t)]

(%i3) diff(z,t); 微分的指令：differ(函數，要微分的變數) //對變數 z 中的 t 變數微分

$$(\%o3) \quad \frac{c\left(\frac{d}{d\,t}\,x\right)}{y} - \frac{c\,x\left(\frac{d}{d\,t}\,y\right)}{y^2}$$

Find the rate of increase in the price z when $x = 1000000, y = 10000$.

$$\frac{\partial z}{\partial x} = \frac{c}{y}, \quad \frac{\partial z}{\partial y} = -\frac{cx}{y^2}.$$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\frac{c}{y}(0.01x)+(-\frac{cx}{y^2})(-\sqrt{x})$$

$$=c\cdot 10^{-4}\cdot 10^{-2}\cdot 10^{6}+c\cdot 10^{6}\cdot (10^{-4})^2\cdot (10^{6})^{1/2}$$

$$=c(1+10)=11c.$$

Example 4.

Use the Chain Rule to compute $\partial z/\partial s$ and $\partial z/\partial t$ where

$z=\frac{x^2}{y}$, $x=st$, $y=s^2-t^2$.

Solution：(%i1) x:st; //建立一變數 st ，名稱叫做 x

(%o1) st

(%i2) y:s^2-t^2; //建立一變數 s^2-t^2 ，名稱叫做 y

(%o2) s^2-t^2

(%i3) z:x^2/y; //建立一變數 $\frac{x^2}{y}$ ，名稱叫做 z

(%o3) $\frac{st^2}{s^2-t^2}$

(%i4) diff(z,s); 微分的指令：differ(函數，要微分的變數) //對變數 z 中的 s 變數微分

(%o4) $-\frac{2\ s\ st^2}{(s^2-t^2)^2}$

(%i5) diff(z,t); 微分的指令：differ(函數，要微分的變數) //對變數 z 中的 t 變數微分

(%o5) $\frac{2\ st^2\ t}{(s^2-t^2)^2}$

$$\frac{\partial z}{\partial x} = \frac{2x}{y}, \quad \frac{\partial z}{\partial y} = -\frac{x^2}{y^2}.$$

$$\frac{\partial x}{\partial s} = t, \quad \frac{\partial y}{\partial s} = 2s.$$

$$\frac{\partial x}{\partial t} = s, \quad \frac{\partial y}{\partial t} = -2t.$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{2x}{y}t - \frac{x^2}{y^2}2s$$

$$= \frac{2st^2}{s^2 - t^2} - \frac{2s^3t^2}{(s^2 - t^2)^2}.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2x}{y}s - \frac{x^2}{y^2}(-2t)$$

$$= \frac{2s^2t}{s^2 - t^2} + \frac{2s^2t^3}{(s^2 - t^2)^2}.$$

As a check, we compute $\partial z/\partial s$ and $\partial z/\partial t$ directly without the Chain Rule.

$$z = \frac{x^2}{y} = \frac{s^2t^2}{s^2 - t^2}.$$

$$\frac{\partial z}{\partial s} = \frac{(s^2 - t^2)2st^2 - (2s)s^2t^2}{(s^2 - t^2)^2} = \frac{2st^2}{s^2 - t^2} - \frac{2s^3t^2}{(s^2 - t^2)^2}.$$

$$\frac{\partial z}{\partial t} = \frac{(s^2 - t^2)(2s^2t) - (2t)s^2t^2}{(s^2 - t^2)^2} = \frac{2s^2t}{s^2 - t^2} + \frac{2s^2t^3}{(s^2 - t^2)^2}.$$

Example 5.

Let z depend on x and y and let $x = r\cos\theta, y = r\sin\theta$. Use the Chain Rule to obtain formulas for $\partial z/\partial r$ and $\partial z/\partial \theta$.

Solution：(%i1) x:r*cos(theta); //建立一變數 $r\cos\theta$ ，名稱叫做 x

```
(%o1) r cos(θ)
```

(%i2) y:r*sin(theta); //建立一變數 $r\sin\theta$ ，名稱叫做 y

```
(%o2) r sin(θ)
```

(%i3) depends(z,[x,y]); 相依的指令：depends(函數，相依的變數) //此題 z 和 x 跟 y 有關係

(%o3) [z(r cos(θ), r sin(θ))]

(%i4) diff(z,r); 微分的指令：differ(函數，要微分的變數) //對變數 z 中的 r 變數微分

$$(\%o4)\quad \sin(\theta)\left(\frac{d}{d\,(r\sin(\theta))}z\right)+\cos(\theta)\left(\frac{d}{d\,(r\cos(\theta))}z\right)$$

(%i5) diff(z,theta); 微分的指令：differ(函數，要微分的變數) //對變數 z 中的 θ 變數微分

$$(\%o5)\quad r\cos(\theta)\left(\frac{d}{d\,(r\sin(\theta))}z\right)-r\sin(\theta)\left(\frac{d}{d\,(r\cos(\theta))}z\right)$$

$$\frac{\partial x}{\partial r}=\cos\theta,\quad \frac{\partial y}{\partial r}=\sin\theta.$$

$$\frac{\partial x}{\partial \theta}=-r\sin\theta,\quad \frac{\partial y}{\partial \theta}=r\cos\theta.$$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta.$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial z}{\partial \theta}=-\frac{\partial z}{\partial x}r\sin\theta+\frac{\partial z}{\partial y}r\cos\theta.$$

Example 6.

A rectangular solid has sides x, y and z. Find the rate of change of the volume

$V = xyz$ if $x=1,\quad y=2,\quad z=3$ (in feet),

$\frac{dx}{dt}=1,\quad \frac{dy}{dt}=-5,\quad \frac{dz}{dt}=2$ (in feet per second).

Solution：(%i1) depends([x,y,z],t); 相依的指令：depends(函數，相依的變數) //此題 x，y 和 z 都和 t 有關係

(%o1) [x(t), y(t), z(t)]

(%i2) V:x*y*z;　//建立一變數 xyz ，名稱叫做 V

(%o2) $x\,y\,z$

(%i3) diff(V,t);　微分的指令：differ(函數，要微分的變數)　//對變數 V 中的 t 變數微分

(%o3) $x\,y\left(\frac{d}{d\,t}\,z\right)+x\left(\frac{d}{d\,t}\,y\right)z+\left(\frac{d}{d\,t}\,x\right)y\,z$

We have $\frac{\partial V}{\partial x}=yz,\quad \frac{\partial V}{\partial y}=xz,\quad \frac{\partial V}{\partial z}=xy$,

so $$\frac{dV}{dt}=\frac{\partial V}{\partial x}\frac{dx}{dt}+\frac{\partial V}{\partial y}\frac{dy}{dt}+\frac{\partial V}{\partial z}\frac{dz}{dt}$$
$$=2\cdot 3\frac{dx}{dt}+1\cdot 3\frac{dy}{dt}+1\cdot 2\frac{dz}{dt}$$
$$=2\cdot 3\cdot 1+1\cdot 3\cdot(-5)+1\cdot 2\cdot 2=-5.$$

Thus the volume is decreasing at -5 cubic feet per second (Figure 11.5.6).

11.6　Implicit Functions

Example 1.

If $z=2x+3y,\quad y=\sin x$,

find $\partial z/\partial x$ and dz/dx when $x=0$.

Solution：(%i1) z:2*x+3*y;　//建立一變數 $2x+3y$ ，名稱叫做 z

(%o1) $3\,y+2\,x$

(%i2) diff(z,x);　微分的指令：differ(函數，要微分的變數)　//對變數 z 中的 x 變數微分

(%o2) 2

(%i3) y:sin(x);　//建立一變數 $\sin x$ ，名稱叫做 y

```
(%o3) sin(x)
```

(%i4) z:2*x+3*y;　//建立一變數 $2x+3y$ ，名稱叫做 z

```
(%o4) 3 sin(x)+2 x
```

(%i5) diff(%,x);　微分的指令：differ(函數，要微分的變數)　//對上式變數 z 中的 x 變數微分

```
(%o5) 3 cos(x)+2
```

(%i6) x:0;　//建立一常數 0，名稱叫做 x

```
(%o6) 0
```

(%i7) 3*cos(x)+2;　//將 x=0 代入 $3\cos x+2$ ，求得值 5

```
(%o7) 5
```

$$\frac{\partial z}{\partial x}=2.$$

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}=2+3\cos x.$$

When $x=0$, $\dfrac{dz}{dx}=2+3\cos 0=5.$

As a check, we find dz/dx directly.

$$z=2x+3y=2x+3\sin x.$$

$$\frac{dz}{dx}=2+3\cos x.$$

When $x=0$, $\dfrac{dz}{dx}=2+3\cos 0=5.$

Example 2.

Use the Chain Rule to obtain a formula for dz/dx where $z=x^y$ and y depends on x.

Solution：(%i1) z:x^y;　//建立一變數 x^y ，名稱叫做 z

(%o1) x^y

(%i2) depends(y,x);　相依的指令：depends(函數，相依的變數)　//此題 y 和 x 有關係

(%o2) [y(x)]

(%i3) diff(z,x);　微分的指令：differ(函數，要微分的變數)　//對變數 z 中的 x 變數微分

(%o3) $x^y\left(\log(x)\left(\frac{d}{d\,x}y\right)+\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x}=yx^{y-1},\quad \frac{\partial z}{\partial y}=(\ln x)x^y.$$

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx}=yx^{y-1}+(\ln x)x^y\frac{dy}{dx}.$$

Example 3.

Find the slope dy/dx of the circle

$$x^2+y^2-4=0$$

at the point $(1,\sqrt{3})$ (see Figure 11.6.5).

Solution：(%i1) dydx(expr,x,y) := -diff(expr,x)/diff(expr,y);　//我們自己定義

$$\frac{dy}{dx}=-\frac{\partial z/\partial x}{\partial z/\partial y}$$

(%o1) $\mathrm{dydx}(expr,x,y):=\frac{-\mathrm{diff}(expr,x)}{\mathrm{diff}(expr,y)}$

(%i2) f:x^2 +y^2-4;　//建立一變數 x^2+y^2-4 ，名稱叫做 f

(%o2) y^2+x^2-4

(%i3) m :dydx(f,x,y);　//將 f 代入第一式所定義的公式中，得到值為斜率 m

(%o3) $-\frac{x}{y}$

(%i4) m : subst([x=1,y=sqrt(3)],m);　//將 x=1，y=$\sqrt{3}$ 代入上式 m 值中，得到斜率

$-\frac{1}{\sqrt{3}}$

(%o4) $-\frac{1}{\sqrt{3}}$

Put $z = x^2 + y^2 - 4 = 0$.

At a point (x, y),

$$\frac{\partial z}{\partial x} = 2x,\quad \frac{\partial z}{\partial y} = 2y,\quad \frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y} = -\frac{x}{y}.$$

At the given point $(1, \sqrt{3})$, $\frac{dy}{dx} = -\frac{1}{\sqrt{3}}$.

In this problem we can solve for y as a function of x and check the answer directly.

$$y = \sqrt{4 - x^2}.$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-2}{2\sqrt{4-1}} = -\frac{1}{\sqrt{3}}.$$

Example 3. (Continued)

Find the equation for the tangent line in Example 3.

Solution：%i1) dydx(expr,x,y) := -diff(expr,x)/diff(expr,y);　//我們自己定義

$$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}$$

(%o1) $\mathrm{dydx}(expr, x, y) := \frac{-\mathrm{diff}(expr, x)}{\mathrm{diff}(expr, y)}$

(%i2) f:x^2 +y^2-4; //建立一變數 x^2+y^2-4 ，名稱叫做 f

(%o2) y^2+x^2-4

(%i3) m :dydx(f,x,y); //將 f 代入第一式所定義的公式中，得到値爲斜率 m

(%o3) $-\frac{x}{y}$

(%i4) m : subst([x=1,y=sqrt(3)],m); //將 x=1，y=$\sqrt{3}$ 代入上式 m 値中，得到斜率

$-\frac{1}{\sqrt{3}}$

(%o4) $-\frac{1}{\sqrt{3}}$

(%i5) [x0,y0] : [1,sqrt(3)]; //設 $x_0=1, y_0=\sqrt{3}$

(%o5) $[1,\sqrt{3}]$

(%i6) tangent : y = m*(x-x0) + y0; //定義斜率的公式爲 $y=m(x-x_0)+y_0$ ，將點 x_0, y_0 代入，即可得切線方程式

(%o6) $y=\sqrt{3}-\frac{x-1}{\sqrt{3}}$

At the point $(1,\sqrt{3})$,

$$\frac{\partial z}{\partial x}=2x=2,\quad \frac{\partial z}{\partial y}=2y=2\sqrt{3},$$

and the tangent line is

$$2(x-1)+2\sqrt{3}(y-\sqrt{3})=0.$$

Example 4.

Find the tangent line and slope of the curve

$y + \ln y + x^3 = 0$

at the point (-1, 1) (Figure 11.6.6).

Solution：(%i1) dydx(expr,x,y) := -diff(expr,x)/diff(expr,y);　//我們自己定義 $\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}$

(%o1) $\mathrm{dydx}(expr, x, y) := \frac{-\mathrm{diff}(expr, x)}{\mathrm{diff}(expr, y)}$

(%i2) f:y+log(y)+x^3;　//建立一變數 $y + \ln y + x^3$ ，名稱叫做 f

(%o2) $\log(y) + y + x^3$

(%i3) m :dydx(f,x,y);　//將 f 代入第一式所定義的公式中，得到值爲斜率 m

(%o3) $-\frac{3x^2}{\frac{1}{y} + 1}$

(%i4) m : subst([x=-1,y=1],m);　//將 x=-1，y=1 代入上式 m 值中，得到斜率 $-\frac{3}{2}$

(%o4) $-\frac{3}{2}$

Put $z = y + \ln y + x^3$.

Then $\frac{\partial z}{\partial x} = 3x^2$, $\frac{\partial z}{\partial y} = 1 + \frac{1}{y}$.

At (-1, 1), $\frac{\partial z}{\partial x} = 3$, $\frac{\partial z}{\partial y} = 2$.

Tangent Line： $3(x+1) + 2(y-1) = 0$.

Slope： $\frac{dy}{dx} = -\frac{3}{2}$.

Example 5.

Find the tangent line and slope of the level curve of the hyperbolic paraboloid

$z = x^2 - y^2$

at the point (a,b) (where $b \neq 0$) (Figure 11.6.7).

Solution：(%i1) dydx(expr,x,y) := -diff(expr,x)/diff(expr,y); //我們自己定義 $\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}$

(%o1) $dydx(expr,x,y) := \frac{-diff(expr,x)}{diff(expr,y)}$

(%i2) z:x^2-y^2; //建立一變數 $x^2 + y^2$ ，名稱叫做 z

(%o2) $x^2 - y^2$

(%i3) m :dydx(z,x,y); //將 f 代入第一式所定義的公式中，得到值爲斜率 m

(%o3) $\frac{x}{y}$

(%i4) m : subst([x=a,y=b],m); //將 x=a，y=b 代入上式 m 值中，得到斜率 $\frac{a}{b}$

(%o4) $\frac{a}{b}$

(%i5) [x0,y0] : [a,b]; //設 $x_0 = a, y_0 = b$

(%o5) $[a, b]$

(%i6) tangent : y = m*(x-x0) + y0; //定義斜率的公式爲 $y = m(x - x_0) + y_0$ ，將點 x_0, y_0 代入，即可得切線方程式

(%o6) $y = \frac{a(x-a)}{b} + b$

The level curve has the equation

$$x^2 - y^2 = a^2 - b^2,$$

$$x^2 - y^2 - (a^2 - b^2) = 0.$$

Put $w = x^2 - y^2 - (a^2 - b^2) = 0$.

Then $\frac{\partial w}{\partial x}=2x,\quad \frac{\partial w}{\partial y}=-2y$.

At (a,b), $\frac{\partial w}{\partial x}=2a,\quad \frac{\partial w}{\partial y}=-2b$.

Tangent Line： $2a(x-a)-2b(x-b)=0$.

Slope： $\frac{dy}{dx}=-\frac{2a}{-2b}=\frac{a}{b}$.

Example 6.

Find $\frac{\partial w}{\partial x}(x,y)$ and $\frac{\partial w}{\partial y}(x,y)$ where

$w=x^2+2y^2+3z^2,\quad z=e^{2x+y}$.

Solution：(%i1) z:(%e)^(5*x+y); //建立一函數 e^{5x+y} ，名稱叫做 z

(%o1) $\%e^{y+5x}$

(%i2) w:x^2+2*(y^2)+3*(z^2); //建立一函數 $x^2+2y^2+3z^2$ ，名稱叫做 w

(%o2) $3\ \%e^{2(y+5x)}+2y^2+x^2$

(%i3) diff(w,x); 微分的指令：differ(函數，要微分的變數) //對函數 w 中的 x 變數微分

(%o3) $30\ \%e^{2(y+5x)}+2x$

(%i4) diff(w,y); 微分的指令：differ(函數，要微分的變數) //對函數 w 中的 y 變數微分

(%o4) $6\ \%e^{2(y+5x)}+4y$

$\frac{\partial w}{\partial x}(x,y,z)=2x,\quad \frac{\partial w}{\partial y}(x,y,z)=4y,\quad \frac{\partial w}{\partial z}(x,y,z)=6z$.

$\frac{\partial z}{\partial x}=5e^{5x+y},\quad \frac{\partial z}{\partial y}=e^{5x+y}$.

Then $\frac{\partial w}{\partial x}(x,y)=2x+6z\cdot 5e^{5x+y}=2x+30ze^{5x+y}$

$$= 2x + 30e^{2(5x+y)}.$$

$$\frac{\partial w}{\partial y}(x, y) = 4y + 6z \cdot e^{5x+y} = 4y + 6e^{2(5x+y)}.$$

Example 7.

Find the tangent plane to the ellipsoid

$x^2 + 2y^2 + 3z^2 = 6$

at the point (1, 1, 1) (see Figure 11.6.10).

Solution：(%i1) dzdx(expr,x,z) := -diff(expr,x)/diff(expr,z); //我們自己定義 $\frac{dz}{dx} = -\frac{\partial F / \partial x}{\partial F / \partial z}$

(%o1) $\mathrm{dzdx}(expr, x, z) := \frac{-\mathrm{diff}(expr, x)}{\mathrm{diff}(expr, z)}$

(%i2) dzdy(expr,y,z) := -diff(expr,y)/diff(expr,z); //我們自己定義 $\frac{dz}{dy} = -\frac{\partial F / \partial y}{\partial F / \partial z}$

(%o2) $\mathrm{dzdy}(expr, y, z) := \frac{-\mathrm{diff}(expr, y)}{\mathrm{diff}(expr, z)}$

(%i3) F:x^2+2*y^2+3*z^2-6; //建立一變數 $x^2 + 2y^2 + 3z^2 - 6$，名稱叫做 F

(%o3) $3\,z^2 + 2\,y^2 + x^2 - 6$

(%i4) m1 :dzdx(F,x,z); //將 F 代入第一式所定義的公式中，得到值爲斜率 m1

(%o4) $-\frac{x}{3\,z}$

(%i5) m2 :dzdy(F,y,z); //將 F 代入第一式所定義的公式中，得到值爲斜率 m2

(%o5) $-\frac{2\,y}{3\,z}$

(%i6) m1: subst([x=1,y=1,z=1],m1);　//將 x=1，y=1，z=1 代入上式 m1 值中，得到斜率 $-\frac{1}{3}$

(%o6) $-\frac{1}{3}$

(%i7) m2: subst([x=1,y=1,z=1],m2);　//將 x=1，y=1，z=1 代入上式 m2 值中，得到斜率 $-\frac{2}{3}$

(%o7) $-\frac{2}{3}$

(%i8) [x0,y0,z0] : [1,1,1];　//設 $x_0=1, y_0=1, z_0=1$

(%o8) [1, 1, 1]

(%i9) tangent : z-z0 = m1*(x-x0) +m2*(y-y0);　//定義斜率的公式為 $z-z_0=m_1(x-x_0)+m_2(y-y_0)$，將點 x_0, y_0, z_0 代入，即可得切線方程式

(%o9) $z-1=-\frac{2(y-1)}{3}-\frac{x-1}{3}$

Put $F(x,y,z)=x^2+2y^2+3z^2-6$.

Then $F_x(x,y,z)=2x,\quad F_y(x,y,z)=4y,\quad F_z(x,y,z)=6z$.

$F_x(1,1,1)=2,\quad F_y(1,1,1)=4,\quad F_z(1,1,1)=6$.

The tangent plane has the equation

$2(x-1)+4(y-1)+6(z-1)=0$.

Find $\partial z/\partial x$ and $\partial z/\partial y$ at (1,1,1).

$$\frac{\partial z}{\partial x}=-\frac{F_x(1,1,1)}{F_z(1,1,1)}=-\frac{2x}{6z}=-\frac{x}{3z},$$
$$\frac{\partial z}{\partial y}=-\frac{F_y(1,1,1)}{F_z(1,1,1)}=-\frac{4y}{6z}=-\frac{2y}{3z}.$$

At (1, 1, 1), $\frac{\partial z}{\partial x}=-\frac{1}{3}$, $\frac{\partial z}{\partial y}=-\frac{2}{3}$.

11.7 Maxima and Minima

Example 1.

Find the maximum and minimum of $z=x^2+y^2-xy-x$ on the closed rectangle $0\le x\le 1, 0\le y\le 1$.

Step 1： The region D is sketched in Figure 11.7.4.

Step 2： $\frac{\partial z}{\partial x}=2x-y-1$, $\frac{\partial z}{\partial y}=2y-x$.

Step 3： $2x-y-1=0$, $2y-x=0$.

Solving for x and y we get one critical point $y=\frac{1}{3}$, $x=\frac{2}{3}$.

The value of z at that point is

$$z=(\frac{2}{3})^2+(\frac{1}{3})^2-\frac{2}{3}\cdot\frac{1}{3}-\frac{2}{3}=-\frac{1}{3}.$$

Step 4： We make a table.

Boundary Line	z	Maximum	Minimum
$x=0$, $0\le y\le 1$	y^2	1 at (0, 1)	0 at (0, 0)
$x=1$, $0\le y\le 1$	y^2-y	0 at corners	$-\frac{1}{4}$ at $(1,\frac{1}{2})$
$y=0$, $0\le x\le 1$	x^2-x	0 at corners	$-\frac{1}{4}$ at $(\frac{1}{4},0)$
$y=1$, $0\le x\le 1$	x^2+1-2x	1 at (0, 1)	0 at (1, 1)

The values from Step3 and 4 are also shown on the sketch of D in Figure 11.7.5.

Example 2.

For a package to be mailed in the United States by parcel post, its length plus its girth (perimeter of cross section) must be at most 84 inches. Find the dimensions of the rectangular box of maximum volume which can be mailed by parcel post.

Step 1 : Let $x, y,$ and z be the dimensions of the box, with z the length. We wish to find the maximum of the volume $V = xyz$

given the side condition

$length + girth = z + 2x + 2y = 84$.

We eliminate z using the side condition and express V as a function of x and y.

$z = 84 - 2x - 2y$,

$V = xy(84 - 2x - 2y)$.

Since $x, y,$ and z cannot be negative the domain is the closed triangle

$0 \le x, \quad 0 \le y, \quad 0 \le 84 - 2x - 2y$.

This is the same as the closed region

$0 \le x \le 42, \quad 0 \le y \le 42 - x$.

The region is sketched in Figure 11.7.6.

Step 2 : $\frac{\partial V}{\partial x} = 84y - 4xy - 2y^2$,

$\frac{\partial V}{\partial y} = 84x - 2x^2 - 4xy$.

Step 3 : $84x - 4xy - 2y^2 = 0$,

$84x - 2x^2 - 4xy = 0$.

Since $x > 0$ and $y > 0$ at all interior points, we have

$84 - 4x - 2y = 0$,

$84 - 2x - 4y = 0$.

There is one critical point

$x = 14, \quad y = 14,$

$V = (84 - 28 - 28) \cdot 14 \cdot 14 = 2(14)^3$.

Step 4： On all three of the boundary lines

$x = 0, \quad y = 0, \quad 84 - 2x - 2y = 0$

we have $V = (84 - 2x - 2y)xy = 0$.

Therefore the maximum value of V on the boundary of D is 0.

Conclusion The maximum of V is at $x = 14, y = 14$, where $V = 2(14)^3$ (Figure 11.7.7). The box has dimensions

$x = 14, \quad y = 14, \quad z = 28.$

Example 3.

Show that the function $z = e^x \ln y$ has no maximum or minimum.

Solution：(%i1) z:(%e)^x*(log(y)); //建立一函數 $e^x \ln y$ ，名稱叫做 z

(%o1) %e^x log(y)

(%i2) diff(z,x); 分的指令：differ(函數，要微分的變數) //對函數 z 的 x 變數微分

(%o2) %e^x log(y)

(%i3) diff(z,y); 分的指令：differ(函數，要微分的變數) //對函數 z 的 y 數微分

(%o3) $\frac{\%e^x}{y}$

(%i4) solve(((%e)^x)/y,y); 方程式指令：solve(方程式，變數) //此題方程式爲 $\frac{e^x}{y}$ ，變數爲 y，得到無解

(%o4) [ ]

The domain is the open region

$-\infty < x < \infty, \quad 0 < y < \infty$.

The partial derivatives are

$$\frac{\partial z}{\partial x} = e^x \ln y, \quad \frac{\partial z}{\partial y} = \frac{e^x}{y}.$$

There are no critical points because $\partial z / \partial y$ is never zero. Therefore there is no maximum or minimum.

Example 4.

Show that the function $z = x^2 + 2y^2$ has no maximum.

Solution：(%i1) z:x^2+2*(y^2); //建立一函數 $x^2 + 2y^2$ ，名稱叫做 z

(%o1) $2\,y^2 + x^2$

(%i2) diff(z,x); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 x 變數微分

(%o2) $2\,x$

(%i3) diff(z,y); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 y 變數微分

(%o3) $4\,y$

(%i4) solve(4*y,y); 方程式指令：solve(方程式，變數) //此題方程式爲 4y，變數爲 y，得到解 y=0

(%o4) [y=0]

(%i5) solve(2*x,x); 方程式指令：solve(方程式，變數) //此題方程式爲 2x，變數爲 x，得到解 x=0

(%o5) [x=0]

The domain is the whole plane.

We have $\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 4y.$

There is one critical point at (0, 0). At this point, $z = 0$. This is not a maximum because, for example, $z = 3$ at (1, 1). Hence z has no maximum.

Notice that z has a minimum at (0, 0) because $x^2 + 2y^2$ is always ≥ 0 (Figure 11.7.9).

Example 5.

Find the point on the plane $4x - 6y + 2z = 7$ which is nearest to the origin.

Solution：(%i1) z:(1/2)*(7-4*x+6*y); //建立一函數 $\frac{1}{2}(7-4x+6y)$，名稱叫做 z

(%o1) $\frac{6y-4x+7}{2}$

(%i2) w:x^2+y^2+z^2; //建立一函數 $x^2 + y^2 + z^2$，名稱叫做 w

(%o2) $\frac{(6y-4x+7)^2}{4}+y^2+x^2$

(%i3) diff(w,x); 微分的指令：differ(函數，要微分的變數) //對函數 w 中的 x 變數微分

(%o3) $2x-2(6y-4x+7)$

(%i4) expand(%); 將式子展開指令：expand(方程式) //展開上式

(%o4) $-12y+10x-14$

(%i5) diff(w,y); 微分的指令：differ(函數，要微分的變數) //對函數 w 中的 y 變數微分

(%o5) $3(6y-4x+7)+2y$

(%i6) expand(%); 將式子展開指令：expand(方程式) //展開上式

(%o6) $20y-12x+21$

(%i7) solve([-12*y+10*x-14,20*y-12*x+21],[x,y]); 方程式指令：solve(方程式，變數) //此題方程式爲$-14+10x-12y=0, 21-12x+20y=0$，變數爲 x，y，得到解$x=\frac{1}{2}, y=-\frac{3}{4}$

(%o7) $[[x=\frac{1}{2}, y=-\frac{3}{4}]]$

Step 1： The distance from the origin to (x,y,z) is $\sqrt{x^2+y^2+z^2}$. It is easier to work with the square of the distance, which has a minimum at the same point that the distance does. So we wish to find the minimum of

$$w=x^2+y^2+z^2$$

given that $4x-6y+2z=7$.

We eliminate z using the plane equation.

$$z=\frac{1}{2}(7-4x+6y),$$

$$w=x^2+y^2+\frac{1}{4}(7-4x+6y)^2.$$

The domain is the whole (x,y) plane.

Step 2： $\frac{\partial w}{\partial x}=2x+2\cdot\frac{1}{4}(-4)(7-4x+6y)=-14+10x-12y$.

$\frac{\partial w}{\partial y}=2y+2\cdot\frac{1}{4}\cdot(6)(7-4x+6y)=21-12x+20y$.

Step 3： $-14+10x-12y=0$,

$21-12x+20y=0$.

Solving for x and y we get one critical point

$$x=\frac{1}{2},\quad y=-\frac{3}{4}.$$

Conclusion We know from geometry that the is a point on the plane which is closest to the origin (the point where a perpendicular line from the origin meets the plane). Therefore w has a minimum and it must be at the critical point

$x=\frac{1}{2},\quad y=-\frac{3}{4}.$

Example 6.

Find the maximum and minimum, if any, of the function

$$z=\frac{1}{(x+y)^2+(x+1)^2+y^2}.$$

Solution：(%i1) z:1/((x+y)^2+(x+1)^2+y^2);　//建立一函數

$\frac{1}{(x+y)^2+(x+1)^2+y^2}$，名稱叫做 z

(%o1) $\frac{1}{(y+x)^2+y^2+(x+1)^2}$

(%i2) diff(z,x);　微分的指令：differ(函數，要微分的變數)　//對函數 z 中的 x 變數微分

(%o2) $-\frac{2(y+x)+2(x+1)}{((y+x)^2+y^2+(x+1)^2)^2}$

(%i3) diff(z,y);　微分的指令：differ(函數，要微分的變數)　//對函數 z 中的 y 變數微分

(%o3) $-\frac{2(y+x)+2y}{((y+x)^2+y^2+(x+1)^2)^2}$

(%i4)
solve([-(2*(y+x)+2*(x+1))/(((y+x)^2+y^2+(x+1)^2))^2,(2*(x+y)+2*y)/(((y+x)^2+y^2+(x+1)^2)^2)],[x,y]);　方程式指令：solve(方程式，變數)　//此題方程式爲

$-\frac{2(y+x)+2(x+1)}{((y+x)^2+y^2+(x+1)^2)^2},\frac{2(y+x)+2y}{((y+x)^2+y^2+(x+1)^2)^2}$，變數爲 x，y，得到解

$x=-\frac{2}{3},y=\frac{1}{3}$

(%o4) $[[x=-\frac{2}{3},y=\frac{1}{3}]]$

Step 1: The domain is the whole (x, y) plane because the denominator is always positive.

Step 2: $\frac{\partial z}{\partial x} = -[2(x+y)+2(x+1)][(x+y)^2+(x+1)^2+y^2]^{-2}$,

$\frac{\partial z}{\partial y} = -[2(x+y)+2y][(x+y)^2+(x+1)^2+y^2]^{-2}$

Step 3: The partial derivatives are zero when

$2(x+y)+2(x+1)=0, \quad 2(x+y)+2y=0$,

or $2x+y+1=0, \quad x+2y=0$.

The critical point is

$x=-\frac{2}{3}, \quad y=\frac{1}{3}$, and $z=3$.

Step 4: Let E be the hyperreal region

$-H \le x \le H, \quad -H \le y \le H$

When H is positive infinite.

Step 5: At a boundary point of E where $x=\pm H, (x+1)^2$ is infinite so z is infinitesimal. At a boundary point where $y=\pm H$, y^2 is infinite so again z is infinitesimal.

Conclusion z has a maximum of 3 at the critical point $(-\frac{2}{3}, \frac{1}{3})$. z has no minimum.

The region E is sketched in Figure 11.7.12.

Example 5.

Find the dimensions of the box of volume one without a top which has the smallest area (if there is one). The box is sketched in Figure 11.7.13.

Solution：(%i1) z:1/(x*y); //建立一函數 $\frac{1}{xy}$ ，名稱叫做 z

(%o1) $\frac{1}{x\,y}$

(%i2) A:x*y+2*x*z+2*y*z; //建立一函數 $xy+\frac{2}{y}+\frac{2}{x}$ ，名稱叫做 A

(%o2) $x\,y+\frac{2}{y}+\frac{2}{x}$

(%i3) diff(A,x); 微分的指令：differ(函數，要微分的變數) //對函數 A 中的 x 變數微分

(%o3) $y-\frac{2}{x^2}$

(%i4) diff(A,y); 微分的指令：differ(函數，要微分的變數) //對函數 A 中的 y 變數微分

(%o4) $x-\frac{2}{y^2}$

(%i5) solve([y-2/(x^2),x-2/(y^2)],[x,y]); 方程式指令：solve(方程式，變數) // 此題方程式爲 $y-\frac{2}{x^2}, x-\frac{2}{y^2}$ ，變數爲 x，y，得到解 $x=\sqrt[3]{2}, y=\sqrt[3]{2}$

(%o5) [[x=1.259921095381759, y=1.259921095381759], [x=1.091123635971721 %i −0.62996052494744, y=1.091123635971721 %i −0.62996052494744], [x=−1.091123635971721 %i − 0.62996052494744, y=−1.091123635971721 %i −0.62996052494744]]

Step 1： Let x, y, and z be the dimensions of the box, with z the height. We want the minimum of the area

$$A = xy + 2xz + 2yz$$

given that

$xyz = 1$.

Eliminating z, we have $z = \frac{1}{xy}$,

$A = xy + \frac{2}{y} + \frac{2}{x}$.

The domain is the open region $x > 0, y > 0$ (see Figure 11.7.14).

Step 2: $\frac{\partial A}{\partial x} = y - \frac{2}{x^2}, \quad \frac{\partial A}{\partial y} = x - \frac{2}{y^2}$.

Step 3: $y - \frac{2}{x^2} = 0, \quad x - \frac{2}{y^2} = 0$.

The critical point is $x = \sqrt[3]{2}, y = \sqrt[3]{2}$,

where $A = 2^{2/3} + 2 \cdot 2^{-1/3} + 2 \cdot 2^{-1/3} = 2^{2/3} + 2^{5/3}$.

Step 4: Take for E the hyperreal region $\varepsilon \le x \le H, \varepsilon \le y \le H$ where ε is positive infinitesimal and H is positive infinite.

Step 5: Let (x, y) be a boundary point of E. As we can see from Figure 11.7.15, there are four possible cases.

Case 1 x is infinitesimal. Then A is infinite because $2/x$ is.

Case 2 y is infinitesimal. A is infinite because $2/y$ is.

Case 3 x is not infinitesimal and y is infinite. A is infinite because xy is.

Case 4 y is not infinitesimal and x is infinite. A is infinite because xy is.

Conclusion A is infinite and hence greater than $2^{2/3} + 2^{5/3}$ on the boundary of E.

Therefore A has a minimum at the critical point $x = \sqrt[3]{2}, y = \sqrt[3]{2}$.

The box has dimensions $x = \sqrt[3]{2}, y = \sqrt[3]{2}, z = \frac{1}{xy} = \frac{1}{\sqrt[3]{4}}$.

11.8 Higher Partial Derivatives

Example 1.

Find the four second partial derivatives of

$z = e^x \sin y + xy^2$.

Solution：(%i1) z:(%e)^x*sin(y)+x*(y^2); //建立一函數 $e^x \sin y + xy^2$ ，名稱叫做 z

(%o1) $\%e^{x} \sin(y) + x\, y^2$

(%i2) diff(z,x); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 x 變數微分

(%o2) $\%e^{x} \sin(y) + y^2$

(%i3) diff(z,y); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 y 變數微分

(%o3) $\%e^{x} \cos(y) + 2\, x\, y$

(%i4) diff(z,x,2); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 x 變數微分 2 次

(%o4) $\%e^{x} \sin(y)$

(%i5) diff(z,y,2); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 y 變數微分 2 次

(%o5) $2\, x - \%e^{x} \sin(y)$

(%i6) diff(%o2,y); 微分的指令：differ(函數，要微分的變數) //對%o2 中的 y 變數微分

(%o6) $\%e^{x} \cos(y) + 2\, y$

(%i7) diff(%o3,x); 微分的指令：differ(函數，要微分的變數) //對%o3 中的 x 變數微分

(%o7) $\%e^{x} \cos(y) + 2\, y$

$\frac{\partial z}{\partial x} = e^x \sin y + y^2$, $\frac{\partial z}{\partial y} = e^x \cos y + 2xy$.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(e^x \sin y + y^2) = e^x \sin y,$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(e^x \cos y + 2xy) = -e^x \sin y + 2x.$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(e^x \sin y + y^2) = e^x \cos y + 2y.$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(e^x \cos y + 2xy) = e^x \cos y + 2y.$$

Notice that in this example the two mixed second partials $\partial^2 z / \partial y \partial x$ and $\partial^2 z / \partial x \partial y$ are equal. The following theorem, shows that it is not just a coincidence.

Example 2.

Find the third partial derivatives of $z = e^{2x} \sin y$.

Solution：(%i1) z:(%e)^(2*x)*sin(y); //建立一函數 $e^{2x} \sin y$ ，名稱叫做 z

(%o1) $\%e^{2x} \sin(y)$

(%i2) diff(z,x); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 x 變數微分

(%o2) $2\,\%e^{2x} \sin(y)$

(%i3) diff(z,y); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 y 變數微分

(%o3) $\%e^{2x} \cos(y)$

(%i4) diff(z,x,2); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 x 變數微分 2 次

(%o4) $4\,\%e^{2x} \sin(y)$

(%i5) diff(z,y,2); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 y 變數微分 2 次

(%o5) $-\%e^{2x} \sin(y)$

(%i6) diff(%o2,y); 微分的指令：differ(函數，要微分的變數) //對%o2 中的 y 變數微分

(%o6) 2 %e^2 x cos(y)

(%i7) diff(%o3,x); 微分的指令：differ(函數，要微分的變數) //對%o3 中的 x 變數微分

(%o7) 2 %e^2 x cos(y)

(%i8) diff(z,x,3); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 x 變數微分 3 次

(%o8) 8 %e^2 x sin(y)

(%i9) diff(z,y,3); 微分的指令：differ(函數，要微分的變數) //對函數 z 中的 y 變數微分 3 次

(%o9) -%e^2 x cos(y)

(%i10) diff(%o4,y); 微分的指令：differ(函數，要微分的變數) //對%o4 中的 y 變數微分

(%o10) 4 %e^2 x cos(y)

(%i11) diff(%o5,x); 微分的指令：differ(函數，要微分的變數) //對%o5 中的 x 變數微分

(%o11) -2 %e^2 x sin(y)

$$\frac{\partial z}{\partial x} = 2e^{2x}\sin y,\quad \frac{\partial z}{\partial y} = e^{2x}\cos y.$$

$$\frac{\partial^2 z}{\partial x^2} = 4e^{2x}\sin y,\quad \frac{\partial^2 z}{\partial y^2} = -e^{2x}\sin y.$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial^2 z}{\partial x\partial y} = 2e^{2x}\cos y,$$

$$\frac{\partial^3 z}{\partial x^3} = 8e^{2x}\sin y,\quad \frac{\partial^3 z}{\partial y^3} = -e^{2x}\cos y.$$

$$\frac{\partial^3 z}{\partial x^2\partial y} = 4e^{2x}\cos y,\quad \frac{\partial^3 z}{\partial x\partial y^2} = -2e^{2x}\sin y.$$

Example 3.

If $z = f(x, y)$ has continuous second partials, $x = r\cos\theta$, and $y = r\sin\theta$, find $\partial^2 z / \partial r^2$.

Solution：(%i1) x:r*cos(theta); //建立一變數 $r\cos\theta$ ，名稱叫做 x

(%o1) $r\cos(\theta)$

(%i2) y:r*sin(theta); //建立一變數 $r\sin\theta$ ，名稱叫做 y

(%o2) $r\sin(\theta)$

(%i3) depends([z],[x,y]); 相依的指令：depends(函數，相依的變數) //此題 z 和 x、y 有關係

(%o3) $[z(r\cos(\theta), r\sin(\theta))]$

(%i4) diff(z,r); 微分的指令：differ(函數，要微分的變數) //對變數 z 中的 r 變數微分

(%o4) $$\sin(\theta)\left(\frac{d}{d\,(r\sin(\theta))}\,z\right)+\cos(\theta)\left(\frac{d}{d\,(r\cos(\theta))}\,z\right)$$

(%i5) diff(z,r,2); 微分的指令：differ(函數，要微分的變數) //對變數 z 中的 r 變數微分 2 次

(%o5) $$\sin(\theta)\left(\sin(\theta)\left(\frac{d^2}{d\left(r^2\sin(\theta)^2\right)}\,z\right)+\cos(\theta)\left(\frac{d^2}{d\,(r\cos(\theta))\,d\,(r\sin(\theta))}\,z\right)\right)+\cos(\theta)\left(\cos(\theta)\left(\frac{d^2}{d\left(r^2\cos(\theta)^2\right)}\,z\right)+\sin(\theta)\left(\frac{d^2}{d\,(r\cos(\theta))\,d\,(r\sin(\theta))}\,z\right)\right)$$

(%i6) expand(%);　//用 expand 指令將上式所得做整理

$$\text{(\%o6)}\quad \sin(\theta)^2\left(\frac{d^2}{d\left(r^2\sin(\theta)^2\right)}z\right)+\cos(\theta)^2\left(\frac{d^2}{d\left(r^2\cos(\theta)^2\right)}z\right)+2\cos(\theta)\sin(\theta)\left(\frac{d^2}{d(r\cos(\theta))\,d(r\sin(\theta))}z\right)$$

We use the Chain Rule three times.

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta\,.$$

$$\frac{\partial^2 z}{\partial r^2}=\frac{\partial}{\partial r}(\cos\theta\frac{\partial z}{\partial x})+\frac{\partial}{\partial r}(\sin\theta\frac{\partial z}{\partial y})$$

$$=\cos\theta\frac{\partial}{\partial r}(\frac{\partial z}{\partial x})+\sin\theta\frac{\partial}{\partial r}(\frac{\partial z}{\partial y})$$

$$=(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r})\cos\theta+(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r})\sin\theta$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+\frac{\partial^2 z}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta$$

$$=\frac{\partial^2 z}{\partial x^2}\cos^2\theta+2\frac{\partial^2 z}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 z}{\partial y^2}\sin^2\theta\,.$$