10.1 VECTOR ALGEBRA

Example 1 Find the components of the vectors represented by the given directed line segments.

(a) $\overrightarrow{(3,2),(5,1)}$.

x-component = 5-3 = 2, y-component = 1-2 = -1.

(b) $\overrightarrow{(0,-2),(2,-3)}$

x-component = 2-0 = 2, y-component = -3-(-2) = -1.

(%i1) A:[3,2];

(%o1) [3,2]

(%i2) B:[5,1];

(%o2) [5,1]

(%i3) B-A;

(%o3) [2,-1]

(%i4) C:[0,-2];

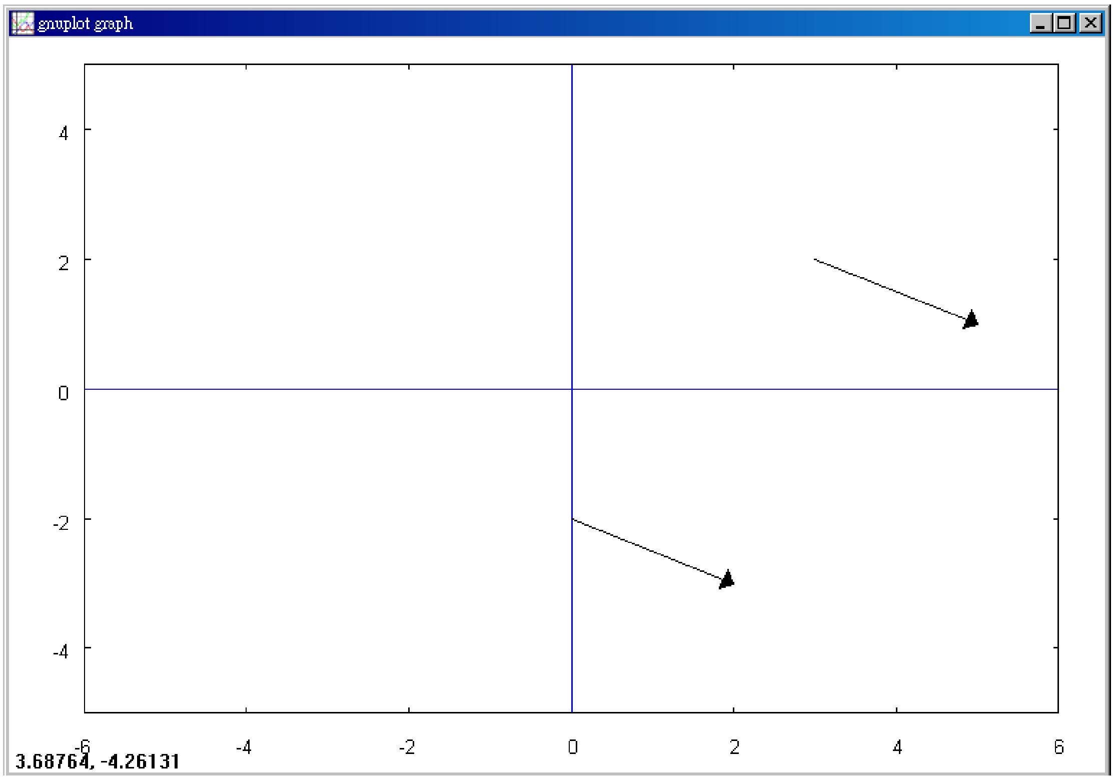
(%o4) [0,-2]

(%i5) D:[2,-3];

(%o5) [2,-3]

(%i6) D-C;

(%o6) [2,-1]



(%i118) load(draw)$
draw2d(xrange = [-6,6],
yrange = [-5,5],

head_length = 0.2,
vector([3,2],[2,-1]),
vector([0,-2],[2,-1]),
line_type = dots,
xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue);

(%o119) [gr2d(vector, vector)]

Example 2 Let **A** be the vector with components -4 and 1, and let P be the point (1, 2). Find Q so that $\overrightarrow{PQ}$ represents **A**.

Q has the x-coordinate 1+(-4) = -3 and the y-coordinate 2+1 = 3. Thus Q = (-3, 3).

http://www.npue.edu.tw/academic/math/index.htm

(%i8) A:[-4,1];

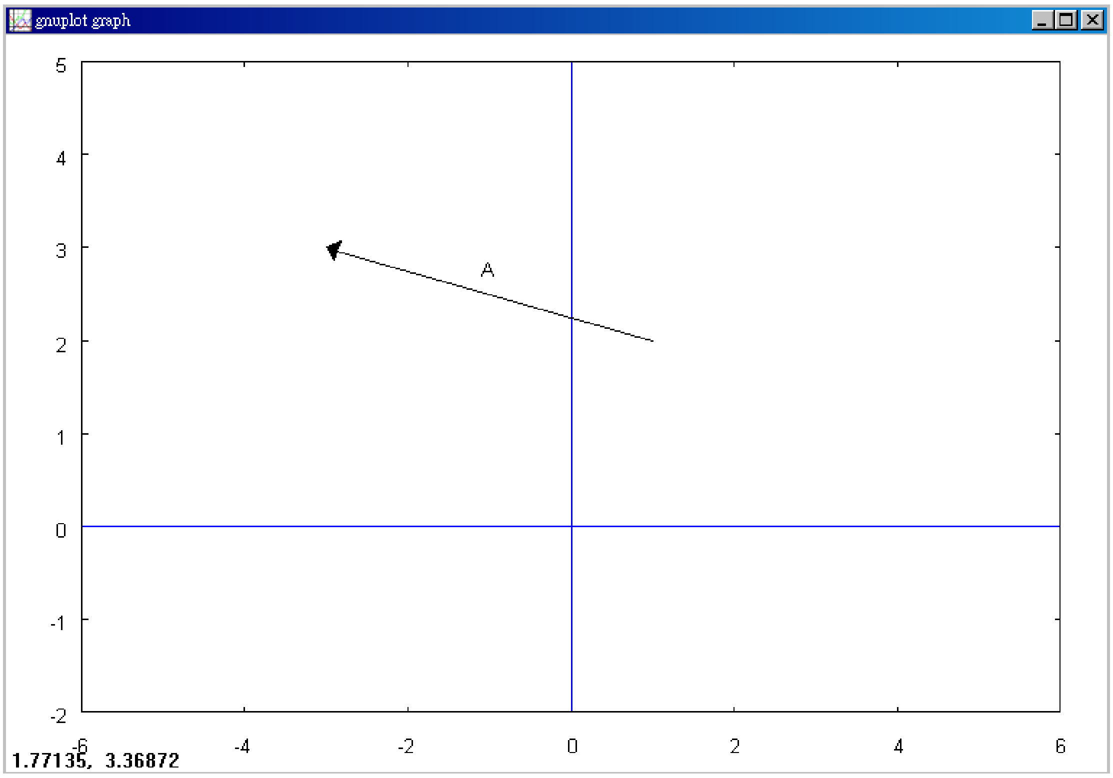
(%o8) [−4 , 1]

(%i10) P:[1,2];

(%o10) [1 , 2]

(%i11) Q:P+A;

(%o11) [−3 , 3]



(%i5) load(draw)$
draw2d(xrange = [-6,6],
yrange = [-2,5],
head_length = 0.2,

vector([1,2],[-4,1]),

line_type = dots,
xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue,label(["A",-1,2.8]));

(%o6) [gr2d(vector, label)]

Example 3 The vector **A** with components 3 and -4 has length

$$|A| = \sqrt{3^2 + (-4)^2} = 5.$$

(%i12) sqrt(3^2+(-4)^2);

(%o12) 5

Example 4 Let **A** = 2**i**–5**j**, **B** = **i**+3**j**.

(a) Find **A**+**B**, **A**-**B**, –**A**, and 6**B**.

A+**B** = (2+1)**i** +(–5+3)**j** = 3**i** –2**j**.

A–**B** = (2–1)**i** +(–5-3)**j** = **i** –8**j**

–**A** = (–1)**A** = (–1)2**i** +(–1)(–5)**j** = –2**i** +5**j**,

6**B** = 6(**i** +3**j**) = 6**i** +18**j**.

(b) Find the vector **D** such that 3**A** + 5**D** = **B.**

$$5\mathbf{D} = -3\mathbf{A} + \mathbf{B},$$

$$\mathbf{D} = \frac{1}{5}(-3\mathbf{A} + \mathbf{B}).$$

$$= \frac{1}{5}(-3\cdot 2 + 1)\mathbf{i} + \frac{1}{5}(-3(-5) + 3)\mathbf{j}.$$

$$= -\mathbf{i} + \frac{18}{5}\mathbf{j}.$$

(%i15) A:[2*i,-5*j];

(%o15) [2 i, -5 j]

(%i16) B:[1*i,3*j];

(%o16) [i, 3 j]

(%i17) A+B;

(%o17) $[3\,i, -2\,j]$

(%i18) A-B;

(%o18) $[i, -8\,j]$

(%i19) -A;

(%o19) $[-2\,i, 5\,j]$

(%i20) 6*B;

(%o20) $[6\,i, 18\,j]$

(%i21) D:1/5*(-3*A+B);

(%o21) $[-i, \frac{18\,j}{5}]$

Example 5 A triangle has vertices (0, 0), (2, -1), and (3 ,1). Find the vectors counterclockwise around the perimeter of the triangle and check that their sum is the zero vector.

The three vectors are

$$\mathbf{A} = (2–0)\mathbf{i} + (–1–0)\mathbf{j} = 2\mathbf{i} – \mathbf{j},$$
$$\mathbf{B} = (3–2)\mathbf{i} + (1–(–1))\mathbf{j} = \mathbf{i} + 2\mathbf{j},$$
$$\mathbf{C} = (0–3)\mathbf{i} + (0–1)\mathbf{j} = –3\mathbf{i} –\mathbf{j}.$$

Their sum is

$$\mathbf{A} + \mathbf{B} + \mathbf{C} =(2 + 1 –3)\mathbf{i} + (–1 + 2 + (–1))\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}.$$

(%i24) A:[2*i,-1*j]-[0,0];

(%o24) [2 i , -j]

(%i25) B:[3*i,1*j]-[2*i,-1*j];

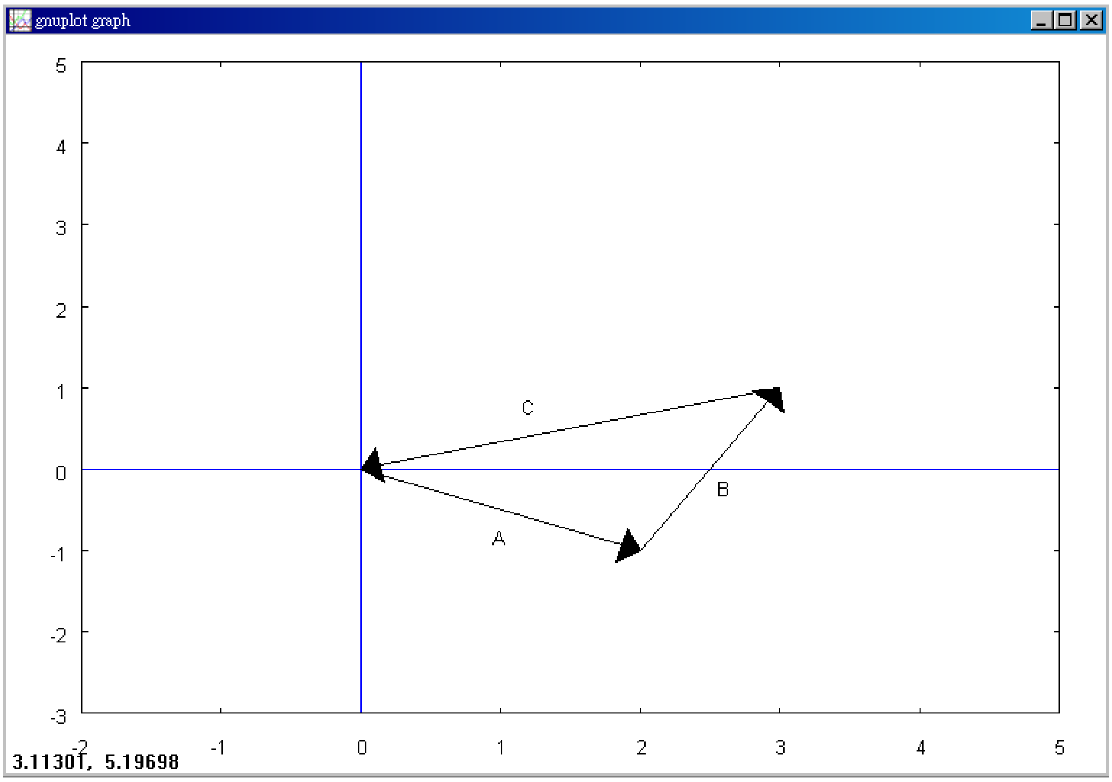
(%o25) [i , 2 j]

(%i26) C:[0,0]-[3*i,1*j];

(%o26) [-3 i , -j]

(%i27) A+B+C;

(%o27) [0 , 0]



```
(%i25) load(draw)$
       draw2d(xrange = [-2,5],

              yrange = [-3,5],

              head_length = 0.2,

              vector([3,1],[ -3,-1]),
              vector([0,0],[2,-1]),
              vector([2,-1],[1,2]),

              line_type = dots,
       xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue,
       label(["A",1,-0.8]),label(["B",2.6,-0.2]),label(["C",1.2,0.8]));
  (%o26) [gr2d(vector,vector,vector,label,label,label)]
```

Example 6 Find the angle between **A** = 3**i** – 4**j** and **B** = **i** + **j.**

$$|A| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5,$$

$$|B| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

$$|B - A| = \sqrt{(3-1)^2 + (-4-1)^2} = \sqrt{4+25} = \sqrt{29},$$

$$\cos\theta = \frac{|A|^2 + |B|^2 - |B-A|^2}{2|A||B|} = \frac{25+2-29}{2\cdot 5\cdot\sqrt{2}}$$

$$= -\frac{2}{10\sqrt{2}} = -\frac{\sqrt{2}}{10}.$$

$$\theta = \arccos\left(-\frac{\sqrt{2}}{10}\right).$$

(%i28) norm(x,y):=sqrt(x^2+y^2);

(%o28) $norm(x,y):=\sqrt{x^2+y^2}$

(%i29) A:norm(3,-4);

(%o29) 5

(%i30) B:norm(1,1);

(%o30) $\sqrt{2}$

(%i32) norm((3-1),(-4-1));

(%o32) $\sqrt{29}$

(%i34) cosP:(25+2-29)/(2*5*sqrt(2));

(%o34) $-\frac{1}{5\sqrt{2}}$

Example 7 Find the unit vector and direction cosines of the given vector.

First find the length., then the unit vector, and then the direction cosines.

(a) $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$ $|A| = \sqrt{2^2 + 1^2} = \sqrt{5}$

Unit vector = $\frac{A}{|A|} = \frac{2i + j}{\sqrt{5}}$

Direction cosines = $\left(\frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}}\right)$

(b) $\mathbf{B} = 5\mathbf{i} - 12\mathbf{j}$ $|B| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$

Unit vector = $\frac{B}{|B|} = \frac{5i - 12j}{13}$

Direction cosines = $(\frac{5}{13}, -\frac{12}{13})$.

(c) $\mathbf{C} = \frac{1}{4}\mathbf{j}$ $\quad |C| = \sqrt{0^2 + (\frac{1}{4})^2} = \frac{1}{4}$

Unit vector = $\frac{\frac{1}{4}j}{\frac{1}{4}} = j$

Direction cosines = (0, 1).

(%i38) norm(x,y):=sqrt(x^2+y^2);

(%o38) $\mathrm{norm}(x,y):=\sqrt{x^2+y^2}$

(%i39) A:[2*i,1*j];

(%o39) $[2\,i, j]$

(%i40) U:A/norm(2,1);

(%o40) $[\frac{2\,i}{\sqrt{5}}, \frac{j}{\sqrt{5}}]$

(%i41) B:[5*i,-12*j];

(%o41) $[5\,i, -12\,j]$

(%i42) B/norm(5,-12);

(%o42) $[\frac{5\,i}{13}, -\frac{12\,j}{13}]$

(%i43) C:[0,j/4];

(%o43) $[0, \frac{j}{4}]$

(%i44) C/norm(0,1/4);

(%o44) $[0, j]$

Example 8 Find the vector A which has length 6 and direction cosines $(-1/2, \sqrt{3}/2)$.

$$\mathbf{A} = 6(-1/2)\mathbf{i} + 6(\sqrt{3}/2)\mathbf{j} = -3\mathbf{i} + 3\sqrt{3}\,\mathbf{j}$$

(%i46) A:[-i/2,j*sqrt(3)/2];

(%o46) $[-\frac{i}{2}, \frac{\sqrt{3}\,j}{2}]$

(%i47) 6*A;

(%o47) $[-3\,i, 3^{3/2}\,j]$

10.2 VECTORS AND PLANE GEOMETRY

Example 1 Find a vector equation for the line through the two points A(2, 1) and B(-4, 0). The vector **D** = **B** – **A** from A to B is given by

$$\mathbf{D} = (-4-2)\mathbf{i} + (0-1)\mathbf{j} = -6\mathbf{i} - \mathbf{j}.$$

Since A is a position vector and D a direction vector of the line, the line has the vector equation

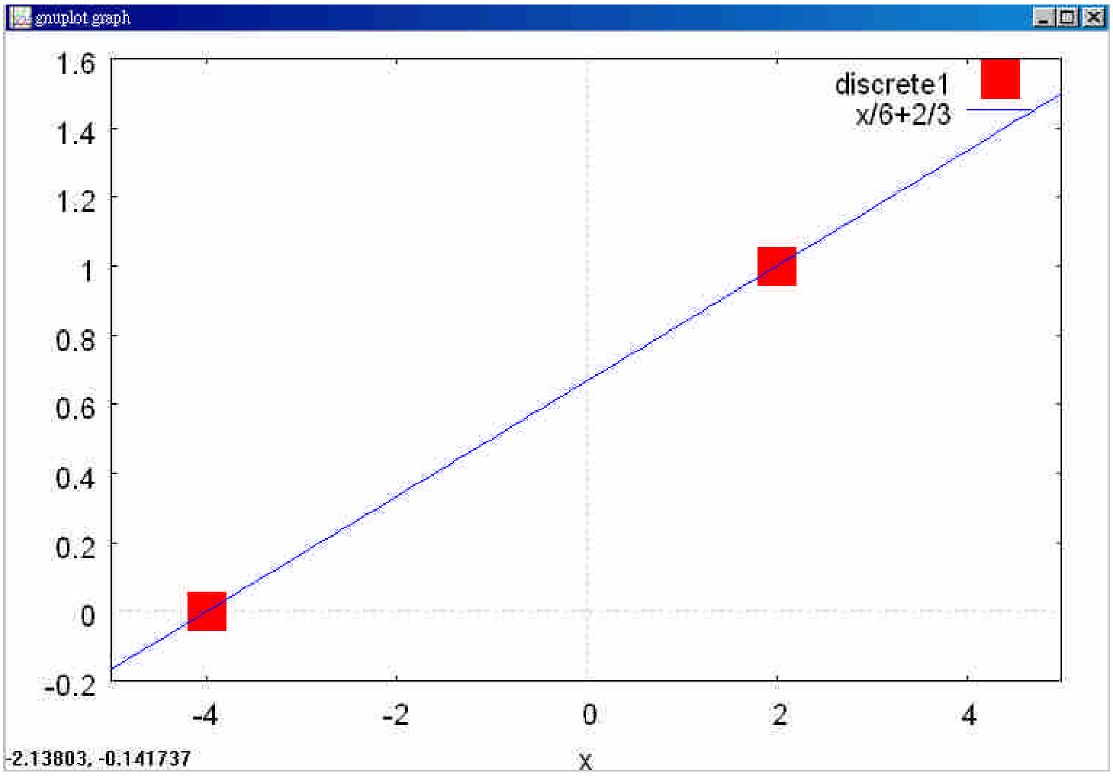
$$\mathbf{X} = \mathbf{A} + t\mathbf{D}$$
$$= 2i + j + t\,(-6i - j).$$

In general, the line L through points A and B has the vector equation **X** = **A** + t(**B-A**) because A is a position vector and **B**–**A** is a direction vector of L.

```
(%i49) A:[2*i,1*j];
(%o49) [2 i , j ]

(%i50) B:[-4*i,0*j];
(%o50) [-4 i , 0 ]

(%i51) D:B-A;
(%o51) [-6 i , -j ]
```



(%i32) xy:[[-4,0],[2,1]]$

(%i33) plot2d([discrete, xy], [style,points])$

(%i34) plot2d([x/6+2/3],[x,-5,5]);

(%o34)

(%i52)

plot2d([[discrete,xy], x/6+2/3], [x,-5,5],
[style, [points,5,2,6], [lines,1,1]]);

(%o52)

Example 2 Find a vector equation for the line in Figure 10.2.8:

$$2x - 3y = 1.$$

Step 1 Find two points on the line by taking two values of x and solving for y.

$$x = 0, \quad 0 - 3y = 1, \quad y = -\frac{1}{3}, \quad (0,-\frac{1}{3})$$

$$x = 1, \quad 2 - 3y = 1, \quad y = \frac{1}{3}, \quad (1,\frac{1}{3})$$

Step 2 Find a position and direction vector.

$$\mathbf{P} = 0\mathbf{i} + (-\frac{1}{3})\mathbf{j} = -\frac{1}{3}\mathbf{j}.$$

$$\mathbf{D} = (1 - 0)\mathbf{i} + (\frac{1}{3} - (-\frac{1}{3}))\mathbf{j} = \mathbf{i} + \frac{2}{3}\mathbf{j}$$

Step 3 Use Theorem 1. The vector equation is

$$\mathbf{X} = \mathbf{P} + t\mathbf{D}$$

$$= -\frac{1}{3}\mathbf{j} + t(\mathbf{i} + \frac{2}{3}\mathbf{j})$$

(%i60) P:[0*i,(-1/3)*j];

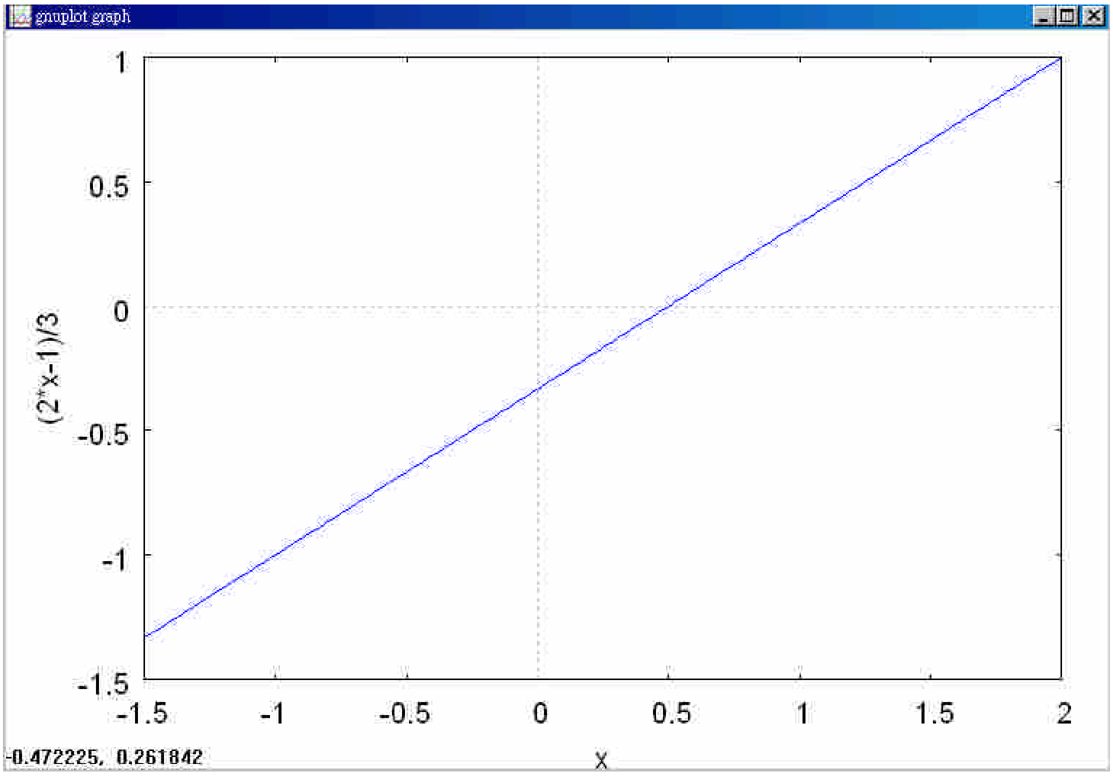
(%o60) $[0, -\frac{j}{3}]$

(%i57) D:[1*i,(1/3)*j]-[0*i,(-1/3)*j];

(%o57) $[i, \frac{2j}{3}]$

(%i61) X:P+t*D;

(%o61) $[i\,t, \frac{2\,j\,t}{3} - \frac{j}{3}]$



(%i63) plot2d([(2*x-1)/3],[x,-1.5,2]);

(%o63)

Example 3 Find a scalar equation for the line in Figure 10.2.9

X = -4i + j + t(i+6j).

First method By Theorem 1,the line has the equation

$$xd_2 - yd_1 = p_1d_2 - p_2d_1,$$
$$6x - y = (-4)\cdot 6 - 1\cdot 1,$$
$$6x - y = -25.$$

Second method We convert the vector equation to parametric equations and then eliminate t.

$$x = -4 + t, \qquad y = 1 + 6t,$$
$$t = x + 4, \qquad y = 1 + 6(x + 4).$$
$$y = 25 + 6x.$$

This is equivalent to the first solution.

(%i69) P:[-4*i,1*j];

(%o69) $[-4\,i, j]$

(%i70) D:[1*i,6*j];

(%o70) $[i, 6\,j]$

(%i71) X:P+t*D;

(%o71) $[i\,t-4\,i, 6\,j\,t+j]$

(%i75) 'x=-4+t;

(%o75) $x=t-4$

(%i76) 'y=1+6*t;

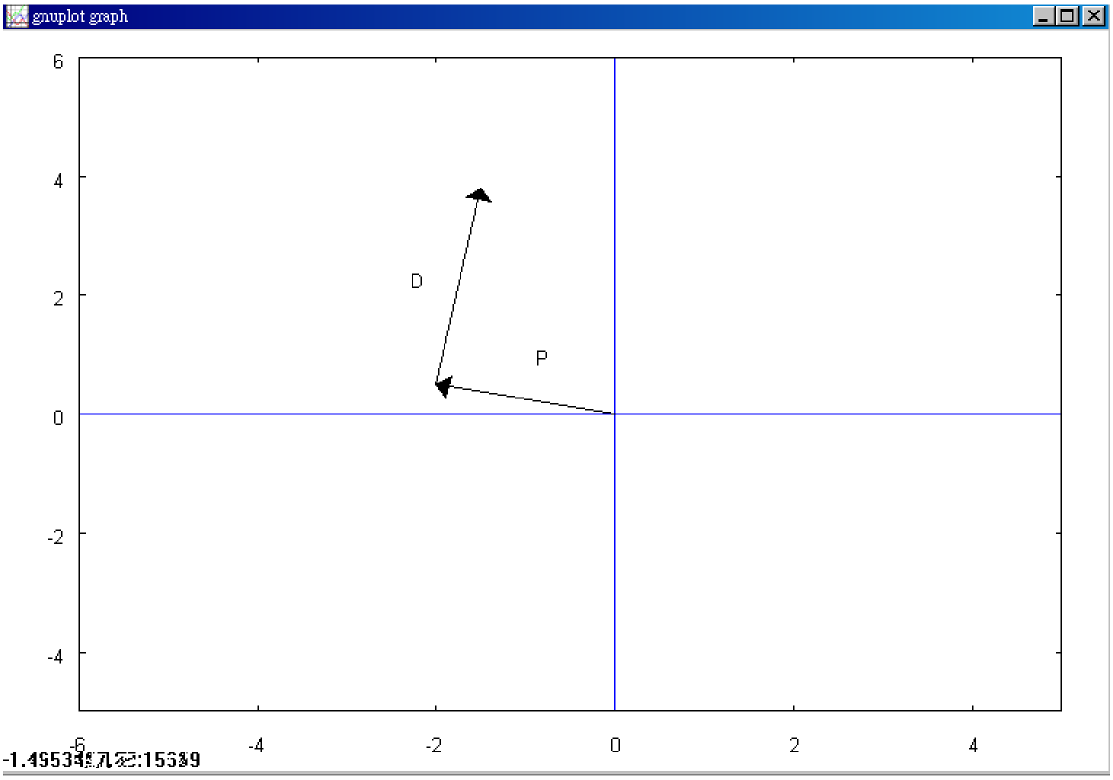
(%o76) $y=6\,t+1$

(%i62) y(t):=1+6*t;

(%o62) $y(t):=1+6\,t$

(%i63) y(x+4);

(%o63) $6\,(x+4)+1$




```
(%i70) load(draw)$
        draw2d(xrange = [-6,5],
               yrange = [-5,6],
               head_length = 0.2,

               vector([0,0],[-2,0.5]),
               vector([-2,0.5],[0.5,3.3]),

               line_type = dots,
        xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue,
        label(["P",-0.8,1]),label(["D",-2.2,2.3]));
 (%o71) [gr2d(vector,vector,label,label)]
```

Example 4 Determine whether the three points

$A(1,3)$, $B(2,5)$, $C(3,10)$

are on the same line.

The line L through A and B has the vector equation

$$\mathbf{X} = \mathbf{A} + t(\mathbf{B} - \mathbf{A})$$

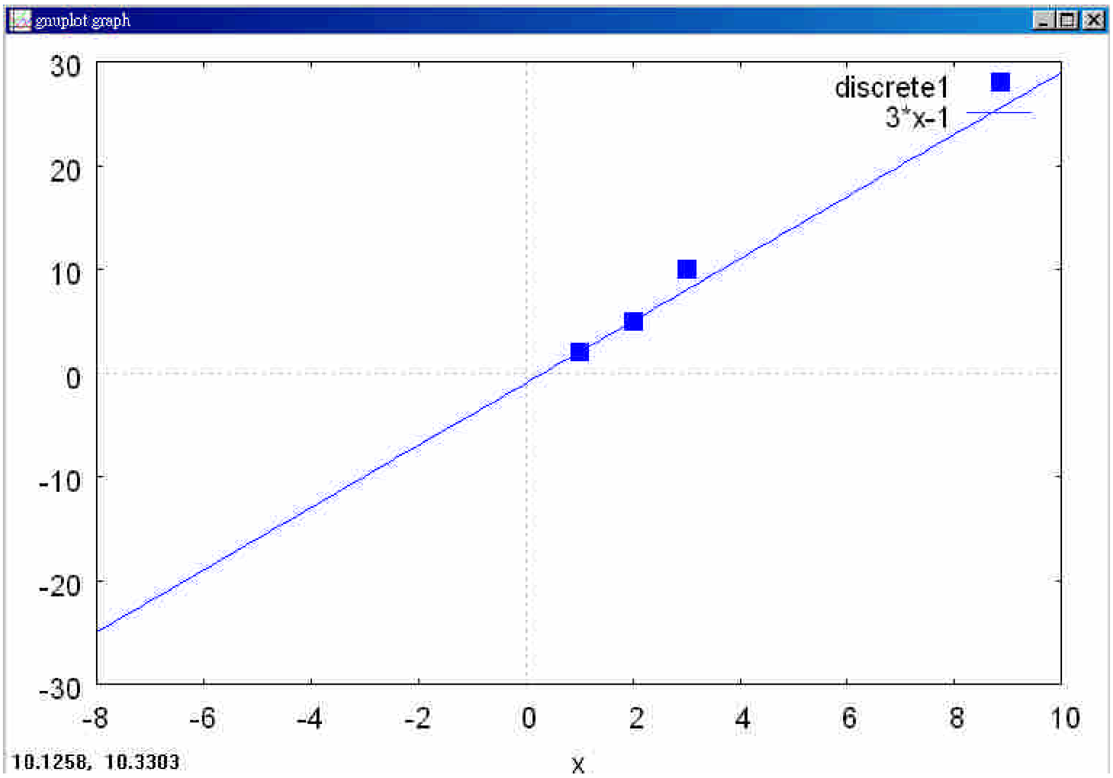
$$= \mathbf{i} + 3\mathbf{j} + t(\mathbf{i} + 2\mathbf{j}) = (1+t)\mathbf{i} + (3+2t)\mathbf{j}.$$

The only point on L with x component 3 is given by

$$3 = 1 + t, \quad t=2, \quad \mathbf{P} = 3\mathbf{i} + 7\mathbf{j}.$$

Since C is another point with x component 3, C is not on L. Therefore A, B, and C are not on the same line, as we see in Figure 10.2.10.

Some applications of vectors to geometry follow.



(%i21) xy:[[3,10],[2,5],[1,2]]$

(%i22) plot2d([discrete, xy], [style,points])$

(%i23) plot2d([3*x-1],[x,-8,10]);

(%o23)

(%i28) plot2d([[discrete,xy], 3*x-1], [x,-8,10],
[style, [points,5,1,6], [lines,1,1]]);

(%o28)

Example 5 Let A and B be two distinct points. Prove that the midpoint of the line segment AB is the point P with position vector $\mathbf{P} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}$.

PROOF We shall prove that the point P is on the line AB and is equidistant from A and B. The line through A and B has the direction vector $\mathbf{D} = \mathbf{B} - \mathbf{A}$. The vector $\mathbf{P}$ has form

$$\mathbf{P} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B} = \mathbf{A} + \frac{1}{2}(\mathbf{B} - \mathbf{A}) = \mathbf{A} + \frac{1}{2}\mathbf{D}.$$

Therefore by Theorem 1, P is on the line AB. To prove that P is equidistant, we show that the vector from A to P is the same as the vector from P to B

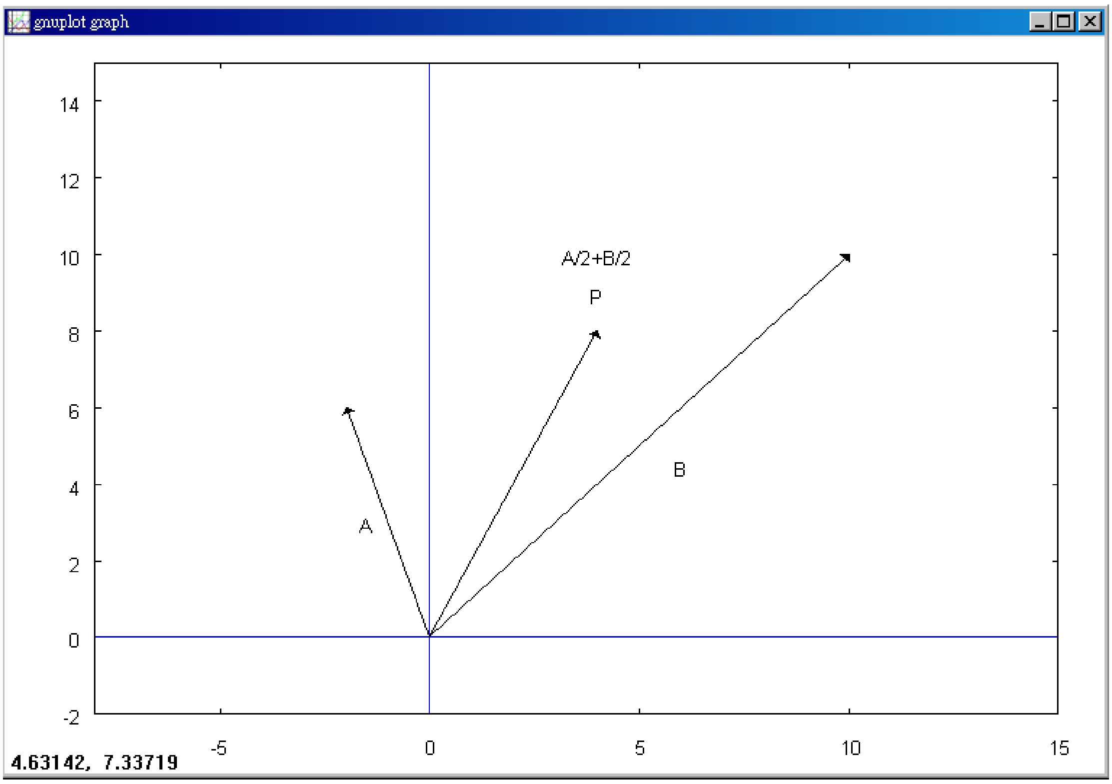
$$\mathbf{P} - \mathbf{A} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B} - \mathbf{A} = \frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{A}.$$

$$\mathbf{B} - \mathbf{P} = \mathbf{B} - \frac{1}{2}\mathbf{A} - \frac{1}{2}\mathbf{B} = \frac{1}{2}\mathbf{B} - \frac{1}{2}\mathbf{A}.$$

```
(%i5) load(draw)$
draw2d(xrange = [-8,15],
yrange = [-2,15],
head_length = 0.2,

vector([0,0],[-2,6]),
vector([0,0],[10,10]),
vector([0,0],[4,8]),

line_type = dots,
xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue,
label(["A",-1.5,3]),label(["B",6,4.5]),label(["P",4,9]),label(["A/2+B/2",4,10]));
```



Example 6 Find the midpoint of the line segment from $A(-1, 2)$ to $B(3,3)$

The points have position vectors

$$\mathbf{A} = -\mathbf{i} + 2\mathbf{j}. \qquad \mathbf{B} = 3\mathbf{i} + 3\mathbf{j}.$$

The midpoint **P** has the position vector

$$\mathbf{P} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B} = \frac{1}{2}(-\mathbf{i} + 2\mathbf{j}) + \frac{1}{2}(3\mathbf{i} + 3\mathbf{j}) = \mathbf{i} + \frac{5}{2}\mathbf{j}.$$

Therefore P is the point $(1, \frac{5}{2})$.

```
(%i1) A:[-1*i,2*j];
(%o1) [-i,2 j]
```

```
(%i2) B:[3*i,3*j];
(%o2) [3 i,3 j]
```

(%i3) P:(A/2)+(B/2);

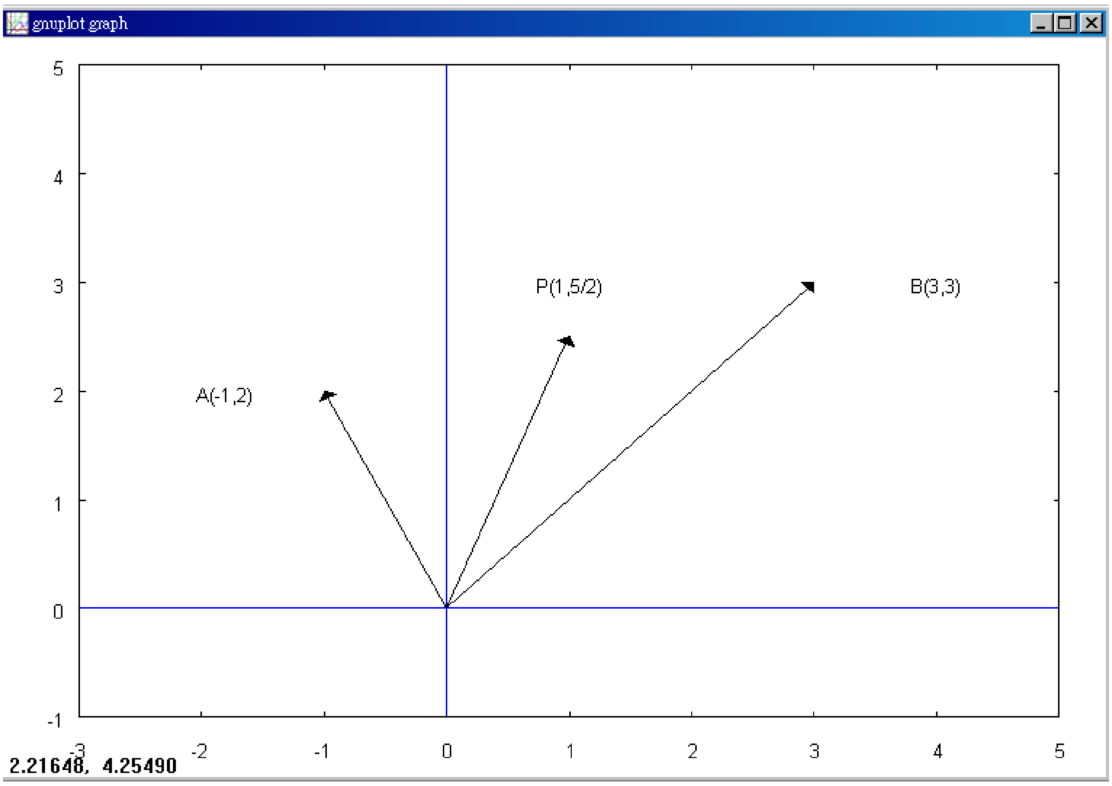
(%o3) $[i, \frac{5\,j}{2}]$

(%i11) load(draw)$
draw2d(xrange = [-3,5],
yrange = [-1,5],
head_length = 0.1,

vector([0,0],[-1,2]),
vector([0,0],[1,5/2]),
vector([0,0],[3,3]),

line_type = dots,
xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue,
label(["A(-1,2)",-1.8,2]),label(["B(3,3)",4,3]),label(["P(1,5/2)",1,3]));

(%o12) [gr2d(vector, vector, vector, label, label, label)]



Example 7 Prove that the diagonals of a parallelogram bisect each other.

PROOF We are given a parallelogram *ABCD*, shown in Figure 10.2.13.

Since the opposite sides represent equal vectors, we have

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D}.$$

The diagonal *AC* has midpoint $\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{C}$ and the other diagonal *BD* has midpoint $\frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{D}$. We show that these two midpoints are equal. The Equation 2 gives

$$\mathbf{C} = \mathbf{B} - \mathbf{A} + \mathbf{D}.$$

Then $\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{C} = \frac{1}{2}\mathbf{A} + \frac{1}{2}(\mathbf{B} - \mathbf{A} + \mathbf{D}) = \frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{D}.$

Thus the two diagonals meet at their midpoints.

Example 8 Prove that the lines from the vertices of a triangle *ABC* to the midpoints of the opposite sides all meet at the single point *P* given by

$$\mathbf{P} = \frac{1}{3}\mathbf{A} + \frac{1}{3}\mathbf{B} + \frac{1}{3}\mathbf{C}.$$

PROOF We are given triangle *ABC*, shown in Figure 10.2.14. Let A', B', C' be the midpoints of the opposite sides. We prove that all three lines AA', BB', CC' pass through the point *P*.

The point A' has position vector

$$\mathbf{A}' = \frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{C}.$$

The line AA' has the direction vector $\mathbf{A}' - \mathbf{A}$. AA' has the vector equation $\mathbf{X} = \mathbf{A} + t(\mathbf{A}' - \mathbf{A})$.

The computation below shows that *P* is on the line AA'

$$\mathbf{P} = \frac{1}{3}\mathbf{A} + \left(\frac{1}{3}\mathbf{B} + \frac{1}{3}\mathbf{C}\right) = \frac{1}{3}\mathbf{A} + \frac{2}{3}\mathbf{A}'$$

$$= \mathbf{A} + \frac{2}{3}(\mathbf{A}' - \mathbf{A}).$$

A similar proof shows that *P* is on BB' and CC'.

10.3 VECTORS AND LINE IN SPACE

Example 1 Given **A** = **i** – **j** + 2**k** and **B** = 2**i** – 2**k**, find **A** + **B**, **A** – **B**, $|A|$, and 3**A**.

A + B = (1 + 2)**i** + (–1 + 0)**j** + (2 – 2)**k** = 3**i** – **j.**
A – **B** = (1 – 2)**i** + (–1 – 0)**j** + (2 – (–2))**k** = –**i** – **j** + 4**k.**

$$|A| = \sqrt{1^2 + (-1)^2 + (2^2)} = \sqrt{6}.$$

3**A** = 3**i** – 3**j** + 6**k**.

(%i5) A:[i,-1*j,2*k];

(%o5) $[i, -j, 2k]$

(%i6) B:[2*i,0,-2*k];

(%o6) $[2i, 0, -2k]$

(%i7) A+B;

(%o7) $[3i, -j, 0]$

(%i8) A-B;

(%o8) $[-i, -j, 4k]$

(%i9) 3*A;

(%o9) $[3i, -3j, 6k]$

(%i10) norm(x,y,Z):=sqrt(x^2+y^2+Z^2);

(%o10) $\mathrm{norm}(x, y, Z) := \sqrt{x^2 + y^2 + Z^2}$

(%i11) norm(1,-1,2);

(%o11) $\sqrt{6}$

Example 2 Find the angle between **A** = **i** – **j** – **k** and **B** = 2**i** + **j** + **k**.

$$|A| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}.$$

$$|B| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}.$$

$$|B - A| = \sqrt{(2-1)^2 + (1-(-1))^2 + (1-(-1))^2}$$

$$= \sqrt{1^2 + 2^2 + 2^2} = 3.$$

$$\cos\theta = \frac{3+6-9}{2\sqrt{3}\sqrt{6}} = 0. \qquad \theta = \arccos 0 = \frac{\pi}{2}.$$

(%i13) A:[i,-1*j,-1*k];

(%o13) $[i, -j, -k]$

(%i14) B:[2*i,j,k];

(%o14) $[2\,i, j, k]$

(%i15) norm(x,y,Z):=sqrt(x^2+y^2+Z^2);

(%o15) norm$(x, y, Z) := \sqrt{x^2 + y^2 + Z^2}$

(%i16) norm(1,-1,-1);

(%o16) $\sqrt{3}$

(%i17) norm(2,1,1);

(%o17) $\sqrt{6}$

(%i18) norm((2-1),(1-(-1)),(1-(-1)));

(%o18) 3

Example 3 Find the unit vectors and direction cosines of the vector **A** = 2**i** + **j**– 2**k.**

We first find the length, then the unit vector, then the direction cosines.

$$|A| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3.$$

$$\mathbf{U} = \frac{A}{|A|} = \frac{2i + j - 2k}{3}.$$

$$\text{Direction cosines} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right).$$

(%i19) A:[2*i,j,-2*k];

(%o19) $[2\,i, j, -2\,k]$

(%i20) norm(2,1,-2);

(%o20) 3

(%i21) U:A/3;

(%o21) $[\frac{2\,i}{3}, \frac{j}{3}, -\frac{2\,k}{3}]$

Example 4 Find a vector equation for the line L with the parametric equations

$$x = 3t + 2. \qquad y = 0t - 4. \qquad z = t + 0.$$

Let **P** = 2**i** – 4 **j,** **D** = 3**i** + **k.**

Then L has the vector equation

X = **P** + t**D** or **X** = (2**i** – 4**j**) + t(3**i** + **k**)

(%i22) P:[2*i,-4*j,0];

(%o22) $[2\,i, -4\,j, 0]$

(%i23) D:[3*i,0,k];

(%o23) [3 i , 0 , k]

(%i25) X:P+t*D;

(%o25) [3 i t+2 i , −4 j , k t]

Example 5 Find a vector equation of the line through the points

A(3 , -4 , 2), B(0 , 8 , 1).

The line has the equation

$\mathbf{X} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} + t((0 - 3)\mathbf{i} + (8- (- 4))\mathbf{j} + (1 - 2)\mathbf{k})$,

$\mathbf{X} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} + t(-3\mathbf{i} + 12\mathbf{j} - \mathbf{k})$.

The formula $\frac{1}{2}(\mathbf{A} + \mathbf{B})$ for the midpoint of the line segment AB holds for three as well as two dimensions.

(%i27) A:[3*i,-4*j,2*k];

(%o27) [3 i , −4 j , 2 k]

(%i28) B:[0,8*j,k];

(%o28) [0 , 8 j , k]

(%i30) X:A+t*(B-A);

(%o30) [3 i −3 i t , 12 j t −4 j , 2 k −k t]

Example 6 Find the midpoint of the line segment AB where

A = (1 , 4 -6), B = (2 , 6 , 0).

The midpoint C has position vector

$$\mathbf{C} = \frac{1}{2}[(\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}) + (2\mathbf{i} + 6\mathbf{j})] = \frac{3}{2}\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

Thus C = $(\frac{3}{2}, 5 , -3)$.

(%i32) A:[i,4*j,-6*k];

(%o32) [i , 4 j , −6 k]

(%i33) B:[2*i,6*j,0];

(%o33) [2 i , 6 j , 0]

(%i34) C:(A+B)/2;

(%o34) $[\frac{3\,i}{2}, 5\,j, -3\,k]$

10.4 PRODUCTS OF VECTORS

Example 1 Compute the inner product of **i**– **j** + 3**k** and **j** + **k**

(i – j + 3k) • (j + k) = 1 • 0 + (-1) • 1 + 3 • 1 = 2.

(%i40) load (eigen)$

(%i43) innerproduct([1,-1,3],[0,1,1]);

(%o43) 2

Example 2 Find the cost of one unit of commodity a, 3 units of commodity b, and 2 unit of commodity c if the prices per unit are 6, 4, and 10 respectively.

cost = (6i + 4j + 10k) • (i + 3j +2k)

= 6 • 1 + 4 • 3 + 10 • 2 = 38.

(%i44) innerproduct([6,4,10],[1,3,2]);

(%o44) 38

Example 3 Suppose a trader buys a commodity vector

A = 40**i** + 60**j** + 100**k**

at the price vector

P = 3**i** + 2**j** + 4**k**

and then sells it at the new price vector

Q = 2**i** + 5**j** + 3**k**.

Find his profit (or loss).

Since the trader pays **P** • **A** and receives **Q** • **A,** his profit is given by

profit = **Q** • **A** – **P** • **A**

Thus profit = (2 • 40 + 5 • 60 + 3 • 100)

–(3 • 40 + 2 • 60 + 4 • 100) = 40.

A positive number indicates a profit and a negative number indicates a loss.

http://www.npue.edu.tw/academic/math/index.htm

(%i52) load (eigen)$

(%i55) F:innerproduct([2,5,3],[40,60,100]);

(%o55) 680

(%i56) G:innerproduct([3,2,4],[40,60,100]);

(%o56) 640

(%i57) F-G;

(%o57) 40

Example 4 A buyer has $7500 and plans to buy a commodity vector **B** in the direction of the unit vector

$$\mathbf{U} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}.$$

Find the largest such commodity vector **B** which he can buy if the price vector is

$$\mathbf{P} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}.$$

We must have **B** = t**U** for some positive t, and also

$$\mathbf{P} \bullet \mathbf{B} = 7500.$$

We solve for t.

$$7500 = \mathbf{P} \bullet \mathbf{B} = \mathbf{P} \bullet t\mathbf{U} = t(\mathbf{P} \bullet \mathbf{U}).$$

$$t = \frac{7500}{P \cdot U} = \frac{7500}{2\cdot\frac{2}{3}+5\cdot\frac{2}{3}+1\cdot\frac{1}{3}} = \frac{7500}{5} = 1500.$$

Thus $\mathbf{B} = t\mathbf{U} = 1000\mathbf{i} + 1000\mathbf{j} + 500\mathbf{k}$

Example 5 A lawnmower is moved horizontally (in the x direction) a distance of 10 feet. Find the work done if the lawnmower is pushed by a force F where

(a) $|F|$=15pounds, $\theta=30^0$

(b) $F = 8\mathbf{i} - 5\mathbf{j}$ in pounds.

(a) $\cos\theta = \frac{1}{2}\sqrt{3}.\qquad |S| = 10.$

$$W = |F||S|\cos\theta = 15 \bullet 10 \bullet \frac{1}{2}\sqrt{3} = 75\sqrt{3}\text{ ft 1bs.}$$

(b) $W = F \bullet S = 8 \bullet 10 + (-5) \bullet 0 = 80$ ft 1bs.

The angle between two vectors can be easily computed using the inner product.

Example 6 Find the angle between the vectors

$$\mathbf{A} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{B} = -\mathbf{i} + 5\mathbf{j} + \mathbf{k}.$$

$$\cos\theta = \frac{A \bullet B}{|A||B|} = \frac{3(-1) + 1 \bullet 5 + (-1) \bullet 1}{\sqrt{3^2 + 1^2 + 1^2}\sqrt{1^2 + 5^2 + 1^2}} = \frac{1}{\sqrt{11 \bullet 27}}$$

$$\arccos\frac{1}{\sqrt{11 \bullet 27}}$$

Here is a list of algebraic rules for inner products. All the rules are easy to prove in either two or three dimensions.

Example 7 Test for $\mathbf{A} \perp \mathbf{B}$ and $A \parallel B$ using the inner product.

(a) $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{B} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}.$

We compute $A \bullet B$ and $|A||B|$.

$$A \bullet B = -1, \qquad |A||B| = 11.$$

Since $A \bullet B \neq 0$, not $A \perp B$.

Since $A \bullet B \neq \pm|A||B|$, not $A \parallel B$

(b)$A = 2i - \sqrt{3}\,j + k, \qquad B = -\sqrt{8}\,i + \sqrt{6}\,j - \sqrt{2}\,k$

$$A \bullet B = -8\sqrt{2}, \qquad |A||B| = 8\sqrt{2}.$$

Therefore $A \parallel B$

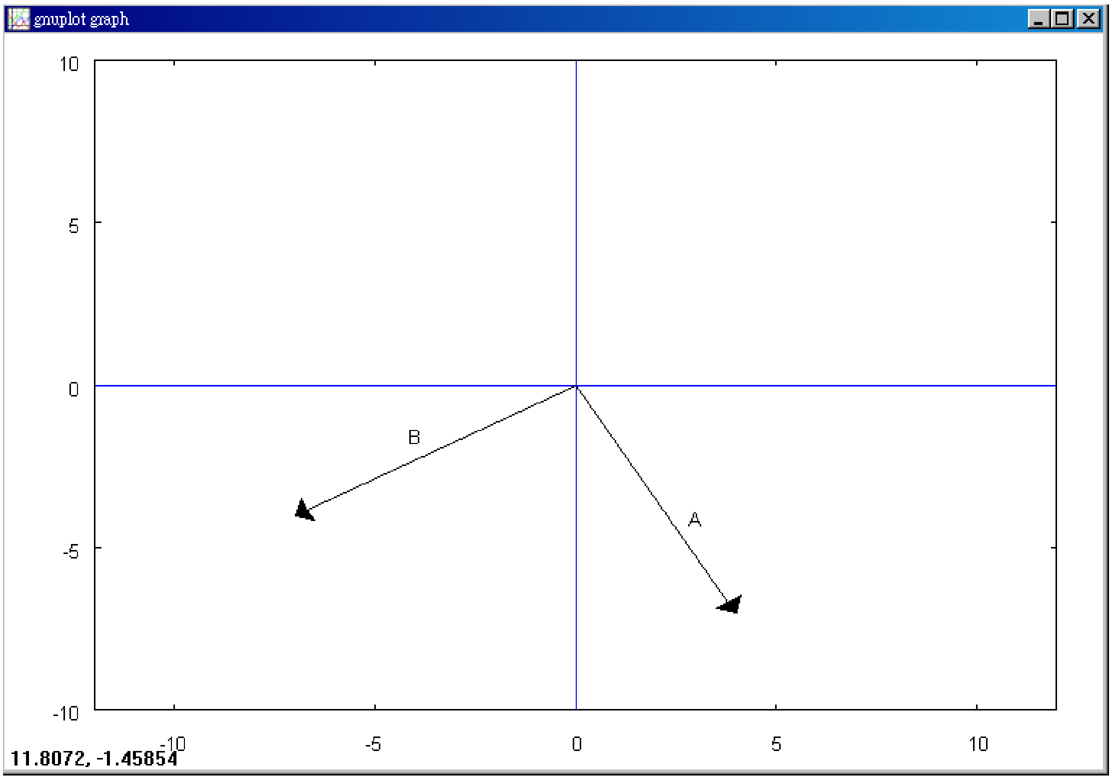
(c) $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{B} = \mathbf{i} - 3\mathbf{j}.$

$A \bullet B = 0.$

Therefore $A \perp B$.

Example 8 Find a vector perpendicular to **A** = 4**i** – 7**j**.

Answer **B** = –7**i** – 4**j.**




```
(%i1) load(draw)$
draw2d(xrange = [-12,12],
        yrange = [-10,10],
        head_length = 0.5,

        vector([0,0],[4,-7]),
        vector([0,0],[-7,-4]),
        line_type = dots,
xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue,label(["B",-4,-1.5]),label(["
A",3,-4]));
 (%o2) [gr2d(vector,vector,label,label)]
```

Example 9 Find $A \times B$ where

$$A = 4i - j + k, \qquad B = 2j - k.$$

$$A \times B = \begin{vmatrix} i & j & k \\ 4 & -1 & 1 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= ((-1)(-1)-1 \bullet 2)i + (1 \bullet 0-4(-1))j + (4 \bullet 2-1(-1) \bullet 0)k$$

$$= -i + 4j + 8k.$$

10.5 **PLANES IN SPACE**

Example 1 (For sketching a plane where a, b ,c and d are nonzero.) Sketch the plane $x + 2y + z = 2$.

Step 1 Find the points where the plane crosses the coordinate axes.

x-axis: When $y = z = 0$, $x = 2$.

The plane crosses the x-axis at (2 , 0 , 0).

y-axis: When $x = z = 0$, $y = 1$.

The plane crosses the y-axis at (0 , 1 , 0).

z-axis: When $x = y = 0$, $z = 2$.

The plane crosses the z-axis at (0 , 0 , 2).

Step 2 Draw the triangle connecting these three points, as shown in Figure 10.5.2. This triangle lies in the plane.

Example 2 (For sketching a plane where two of a, b ,c are nonzero and $d \neq 0$.)

Sketch the plane $2x + z = 4$.

Step 1 Find the points where the plane crosses the x- and z-axes.

The plane crosses the x-axis at (2 , 0 , 0).

The plane crosses the z-axis at (0 , 0 , 4).

Step 2 The plane is parallel to the y-axis. Draw a rectangle with two sides parallel to the y-axis and two sides parallel to the line segment from (2 , 0 , 0) to (0, 0, 4), as in Figure 10.5.3. This rectangle lies in the plane.

Example 3 (For sketching a plane with $d = 0$.) Sketch the plane $x + 2y – z = 0$.

Step 1 The plane passes through the origin because (0, 0, 0) is a solution of the equation. Find another point where x=0 and a third point where y or z=0,

$x=0, \quad y=1, \quad z=2,$

$x=1, \quad y=0, \quad z=1.$

Step 2 Connect the points (0, 0, 0), (0, 1, 2), (1, 0, 1) to form a triangle which lies in the plane, as in Figure 10.5.4.

Example 4 The plane $2x + 3y – z = 5$ has the normal vector

$$\mathbf{N} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

and the vector equation

$$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \bullet \mathbf{X} = 5.$$

Example 5 Find the vector and scalar equations for the plane with position and normal vectors

$$\mathbf{P} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}, \qquad \mathbf{N} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}.$$

We first compute $\mathbf{N} \bullet \mathbf{P}$,

$$\mathbf{N} \bullet \mathbf{P} = 1 \bullet 3 + 1 \bullet (-1) + 4 \bullet (-2) = -6.$$

A vector equation is $(\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \bullet \mathbf{X} = -6$.

A scalar equation is $x + y + 4z = -6.$

Example 6 Find the plane with position vector $\mathbf{P} = \mathbf{k}$ and direction vectors $\mathbf{C} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{D} = -\mathbf{j}$.

First we find a normal vector of the plane,

$\mathbf{N} = \mathbf{C} \times \mathbf{D} = (1 \bullet 0 - 1 \bullet (-1))\mathbf{i} + (1 \bullet 0 - (-2) \bullet 0)\mathbf{j} + ((-2)(-1) - 1 \bullet 0)\mathbf{k}$

$= \mathbf{i} + 2\mathbf{k}.$

Then $\mathbf{N} \bullet \mathbf{P} = 1 \bullet 0 + 0 \bullet 0 + 2 \bullet 1 = 2.$

The plane has the vector equation $(\mathbf{i} + 2\mathbf{k}) \bullet \mathbf{X} = 2$

and the scalar equation $x + 2z = 2.$

Example 7 Find the plane through the three points

$P(-1 , 3 , 1)$, $Q(1 , 2, 3)$, $S(-1 , -1 , 0)$.

The plane has position vector

$\mathbf{P} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

and the two direction vectors

$\mathbf{C} = \mathbf{Q} - \mathbf{P} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k},$

$\mathbf{D} = \mathbf{S} - \mathbf{P} = -4\mathbf{j} - \mathbf{k}.$

A normal vector of the plane is

$\mathbf{N} = \mathbf{C} \times \mathbf{D} = ((-1)(-1) - 2(-4))\mathbf{i} + (2 \bullet 0 - 2(-1))\mathbf{j} + (2(-4) - (-1) \bullet 0)\mathbf{k}$

$= 9\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}.$

Then $\mathbf{N} \bullet \mathbf{P} = 9(-1) + 2 \bullet 3 + (-8) \bullet 1 = -11.$

The plane has the vector equation $(9\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}) \bullet \mathbf{X} = -11$

and the scalar equation $9x + 2y - 8z = -11$

Example 8 Determine whether the plane $3x - 2y + z = 4$ and the line $\mathbf{X} = (3\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + \mathbf{j} - \mathbf{k})$ are parallel.

The plane has the normal vector $\mathbf{N} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

The line has the direction vector $\mathbf{D} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

We compute $\mathbf{N} \bullet \mathbf{D} = 3 \bullet 1 + (-2) \bullet 1 + 1(-1) = 0$

Example 9 Find the line L through the point P(1 , 2 , 3) which is perpendicular to the plane $3x - 4y + z = 10$.

The plane has the normal vector $\mathbf{N} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

Therefore $\mathbf{N}$ is a direction vector of L, and L has the vector equation

$\mathbf{X} = \mathbf{P} + t\mathbf{N},$

$$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$$

Example 10 Find the plane p containing the line $\mathbf{X} = \mathbf{i} + t(\mathbf{j} + \mathbf{k})$ which is perpendicular to the plane $x + 3y - 2z = 0$.

The given plane q has the normal vector $\mathbf{M} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

and the given L has the direction vector **N** which is perpendicular to both **M** and **D**, so we take

$$\mathbf{N} = \mathbf{M} \times \mathbf{D} = \begin{vmatrix} i & j & k \\ 1 & 3 & -2 \\ 0 & 1 & 1 \end{vmatrix}, \qquad \mathbf{N} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$$

The vector $\mathbf{P} = \mathbf{i}$ is a position vector of L and therefore a position vector of p

So p has the vector equation

$$\mathbf{N} \bullet \mathbf{X} = \mathbf{N} \bullet \mathbf{P}$$

$$(5\mathbf{i} - \mathbf{j} + \mathbf{k}) \bullet \mathbf{X} = 5,$$

and the scalar equation $5x - y + z = 5$

Example 11 Find the point at which the line $\mathbf{X} = \mathbf{i} - \mathbf{j} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j} - \mathbf{k})$ intersects the plane $3x - 2y + z = 4$.

The line has the parametric equations

$$x = 1 + 3t, \qquad y = -1 - t, \qquad z = 1 - t.$$

We substitute these in the equation for the plane and solve for t.

$$3(1 + 3t) - 2(-1 - t) + (1 - t) = 4,$$

$$6 + 10t = 4,$$

$$t = -\frac{1}{5}$$

Therefore the point of intersection is given by the parametric equations for the line at $t = -\frac{1}{5}$.

$$x = \frac{2}{5}, \qquad y = -\frac{4}{5}, \qquad z = \frac{6}{5},$$

Example 12 Find the line L of intersection of the planes

$$4x - 5y + z = 2,$$

$$x + 2z = 0.$$

Step 1 To get a position vector of L, we find any point on both planes. Setting $z = 0$

and solving for x and y, we obtain the point S(0, $-\frac{2}{5}$, 0) on both planes.

Thus **S** = $-\frac{2}{5}\mathbf{j}$ is a position vector of L.

Step 2 To get a direction vector **D** of L we need a vector perpendicular to the normal vector of both planes. The normal vectors are

$$\mathbf{M} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}, \quad \mathbf{N} = \mathbf{i} + 2\mathbf{k}$$

We take $$\mathbf{D} = \mathbf{M} \times \mathbf{N} = \begin{vmatrix} i & j & k \\ 4 & -5 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= -10\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}.$$

Thus L is the line $\mathbf{X} = -\frac{2}{5}\mathbf{j} + t(-10\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$.

10.6 VECTOR VALUED FUNCTIONS

Example 1 Find the vector equation for a particle which moves counterclockwise around the unit circle, and is at the point (1 , 0) at time t=0

The motion is given by the parametric equations

$$x = \cos t, \qquad y = \sin t,$$

and the vector equation $\mathbf{X} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}.$

Example 2 A ball thrown at time t=0 with initial velocity of v_1 in the x direction and v_2 in the y direction will follow the parabolic curve

$$x = v_1 t, \qquad y = v_2 t - 16t^2$$

The curve has the vector equation $\mathbf{X} = v_1 t\,\mathbf{i} + (v_2 t - 16t^2)\mathbf{j}.$

Example 3 A point on the rim of a wheel rolling along a line traces out a curve called *a cycloid*. Find the vector equation for the cycloid if the wheel has radius one, rolls at one radian per second along the x-axis, and starts at t=0 with the point at the origin.

As we can see from the close-up in Figure 10.6.3, the parametric equations are $x = t - \sin t, \qquad y = 1 - \cos t.$

The vector equation is $\mathbf{X} = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}.$

Example 4 The space curve

$$\mathbf{X} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

is *a circular helix*. The point (x , y) goes around a horizontal circle of radius one whose center is rising vertically at a constant rate.

Example 5 In economics the price vector may change with time and thus be a vector valued function of t. Find the price vector function **P**(t) for three commodities such that the first commodity has price t^2, the second has price $t+1$, and the price of the third commodity is the sum of the other two ($t \ge 0$). The answer is

$$\mathbf{P}(t) = t^2\,\mathbf{i} + (t+1)\mathbf{j} + (t^2 + t + 1)\mathbf{k}.$$

10.7 VECTOR DERIVATIVES

Example 1 Find dX/dt where

$$X = t^{1/3}\mathbf{i} + \frac{1}{t+1}\mathbf{j} + 2t\,\mathbf{k}, \qquad t \neq -1.$$

$$dX/dt = \frac{1}{3}t^{-2/3}\mathbf{i} - (t+1)^{-2}\mathbf{j} + 2\mathbf{k}.$$

dX/dt is undefined at $t = 0.$ and $t = -1.$

Example 2 Find the vector equation of the tangent line for the spiral

$$F(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{1}{4}t\mathbf{k}$$

at the point $t = \pi/3$.

The derivative is $F'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \frac{1}{4}t\mathbf{k}$

At $t = \pi/3$ the tangent line has the equation

$$X = F(\pi/3) + t\,F'(\pi/3)$$

or

$$X = (\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \frac{\pi}{12}\mathbf{k}) + t\,(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{4}\mathbf{k}).$$

Example 3 Find the length of the helix

$$X = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{1}{4}t\mathbf{k},$$

From t = a to t = b.

$$\frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t, \qquad \frac{dz}{dt} = \frac{1}{4},$$

$$s = \int_a^b \sqrt{\sin^2 t + \cos^2 t + \frac{1}{16}}\,dt$$

$$= \int_a^b \sqrt{1 + \frac{1}{16}}\,dt = \int_a^b \frac{\sqrt{17}}{4}\,dt = \frac{\sqrt{17}}{4}(b-a).$$

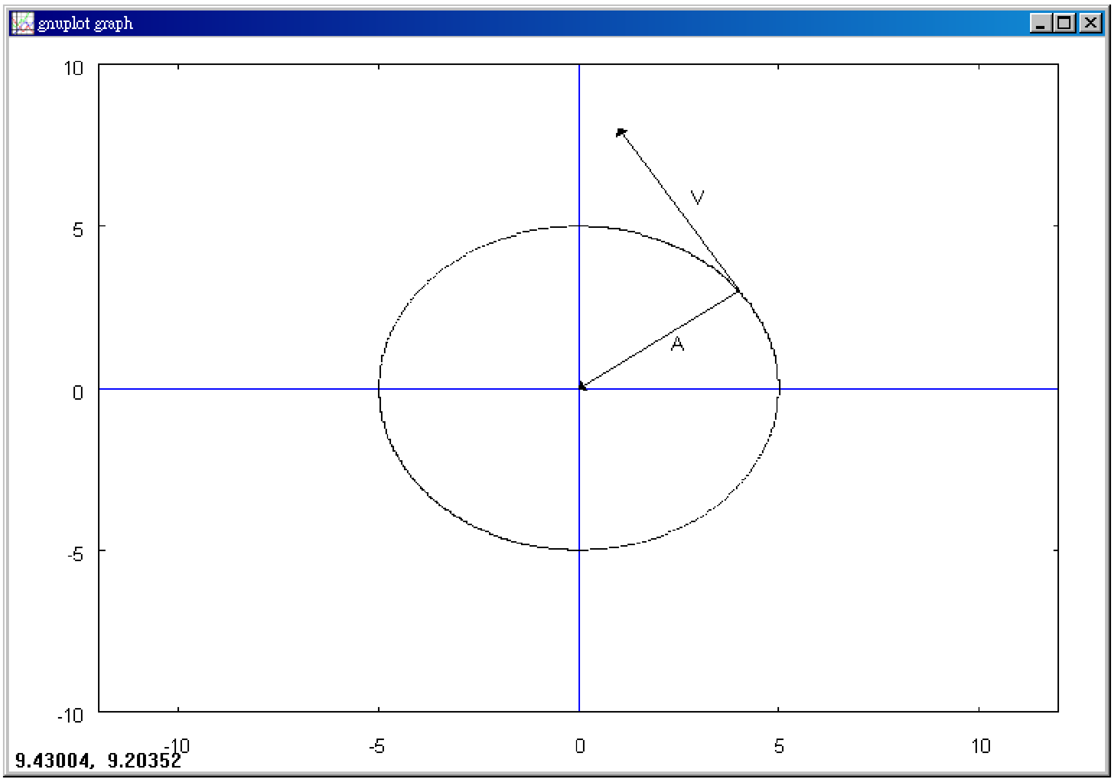
Example 4 Find the velocity, speed, and acceleration of a particle which moves around the unit circle with position vector

$$S = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}.$$

Velocity: $V = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

Speed: $|V| = \sqrt{\sin^2 t + \cos^2 t} = 1$

Acceleration: $A = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$



(%i25) load(draw)$

```
draw2d(implicit(x^2=25-y^2,x,-6,6,y,-6,6),xrange = [-12,12],

    yrange = [-10,10],

    head_length = 0.2,

    vector([4,3],[-4,-3]),
    vector([4,3],[-3,5]),

    line_type = dots,
xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue,
label(["V",3,6]),label(["A",2.5,1.5]));
```

(%o26) [gr2d(implicit, vector, vector, label, label)]

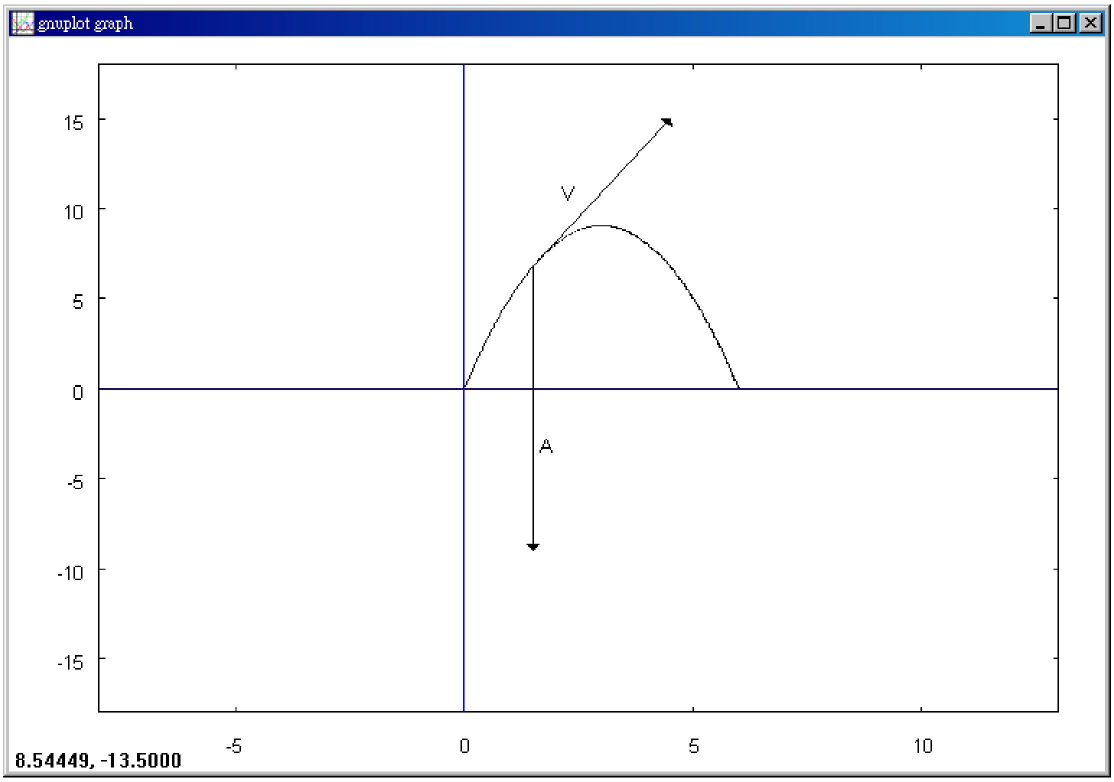
Example 5 Find the velocity, speed, and acceleration of a ball moving on the parabolic curve

$$S = v_1 t\,\mathbf{i} + (v_2 t - 16t^2)\mathbf{j}.$$

Velocity: $V = v_1 t\,\mathbf{i} + (v_2 - 32t)\mathbf{j}$

Speed: $|V| = \sqrt{v_1^2 + (v_2 - 32t)^2}$

Acceleration: $A = -32\mathbf{j}.$




```
(%i64) load(draw)$
draw2d(implicit(y=-(x-3)^2+9,x,0,10,y,0,20),xrange = [-8,13],

        yrange = [-18,18],

        head_length = 0.2,

        vector([1.5,6.75],[3,8.25]),
        vector([1.5,6.75],[0,-15.75]),

        line_type = dots,
```

xaxis=true,xaxis_color=blue,yaxis=true,yaxis_color=blue,
label(["V",2.3,11]),label(["A",1.8,-3]));

```
(%o65) [gr2d(implicit,vector,vector,label,label)]
```

Example 6 Find the position vector of a particle which moves with velocity

$$V = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\cos t\,\mathbf{k}$$

and at time $t=0$ has position F(0) = **i** + 2**k**

we find each component separately by integration.

$$f_1'(t) = -\sin t, \qquad f_1(0)=1$$

$$f_1(t) = \cos t + C_1.$$

$$1 = \cos t + C_1., \qquad C_1 = 0$$

$$f_1(t) = \cos t.$$

$$f_2'(t) = \cos t, \qquad f_2(0)=0$$

$$f_2(t) = \sin t + C_2.$$

$$0 = \sin 0 + C_2., \qquad C_2 = 0$$

$$f_2(t) = \sin t..$$

$$f_3'(t) = \sin t\cos t, \qquad f_3(0)=2.$$

$$f_3(t) = \frac{1}{2}\sin^2 t + C_3,$$

$$2 = \frac{1}{2}\sin^2 0 + C_3, \qquad C_3 = 2.$$

$$f_3(t) = \frac{1}{2}\sin^2 t + 2.$$

$$F(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (\frac{1}{2}\sin^2 t + 2)\mathbf{k}.$$

10.8 **HYPERREAL VECTORS**

Example 1 Let ε be a positive infinitesimal and H be a positive infinite hyperreal number. The vector $5\varepsilon\,\mathbf{i} + \varepsilon^2\mathbf{k}$ is infinitesimal. Its length is

$$\sqrt{25\varepsilon^2 + 0 + \varepsilon^4} = \varepsilon\sqrt{25+\varepsilon^2} \approx 0.$$

The vector $\varepsilon\,\mathbf{i} + \mathbf{j} + \mathbf{k}$ is finite but not infinitesimal. Its length is

$$\sqrt{\varepsilon^2 + 1^2 + 1^2} = \sqrt{\varepsilon^2 + 2} \approx \sqrt{2}.$$

The vector $\mathbf{i} + \varepsilon\,\mathbf{j} + H\mathbf{k}$ is infinite. Its length is

$$\sqrt{1+\varepsilon^2 + H^2} > H.$$

Example 2 Here are some vectors of type (b), (c), and (d)

(b) The vector $\mathbf{B} = \sin\varepsilon\,\mathbf{i} + \cos\varepsilon\,\mathbf{j}$ has real length but nonreal direction (where ε is a positive infinitesimal). **B** has length one.

$$|B| = \sqrt{\sin^2\varepsilon + \cos^2\varepsilon} = 1.$$

However, B is its own unit vector and is not real, so it has nonreal direction.

(c) The following vectors have nonreal lengths but real directions.

$3\varepsilon\,\mathbf{i} + 4\varepsilon\,\mathbf{j}$, infinitesimal length 5ε,

$(6 + 3\varepsilon)\mathbf{i} + (8 + 4\varepsilon)\mathbf{j}$, finite length $5(2+\varepsilon)$,

$3H\mathbf{i} + 4H\mathbf{j}$, infinite length 5H.

All three of these vectors are parallel and have the same real unit vector

$$\mathbf{U} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

(d) The vector $\mathbf{D} = \mathbf{i} + \varepsilon\,\mathbf{j}$ has nonreal length and nonreal direction. Its length is $\sqrt{1+\varepsilon^2}$, and its unit vector is

$$\mathbf{U} = \frac{1}{\sqrt{1+\varepsilon^2}}\mathbf{i} + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}\mathbf{j}.$$

Example 3   The vectors

$$\mathbf{A} = 2\mathbf{i}, \quad \mathbf{B}=2\mathbf{i} + \varepsilon\,\mathbf{j}, \quad \mathbf{C}=-\varepsilon\,\mathbf{i} + \varepsilon^2\,\mathbf{j}$$

are almost parallel to each other. Their unit vectors are **i** ,

$$\frac{2}{\sqrt{4+\varepsilon^2}}\mathbf{i} + \frac{\varepsilon}{\sqrt{4+\varepsilon^2}}\mathbf{j} \approx \mathbf{i}, \qquad \frac{-\varepsilon}{\sqrt{\varepsilon^2+\varepsilon^4}}\mathbf{i} + \frac{\varepsilon^2}{\sqrt{\varepsilon^2+\varepsilon^4}}\mathbf{j} \approx -\mathbf{i}.$$