* 微分：瞬間改變的比率

$$\underbrace{\lim_{x_2\to x_1}}_{\text{瞬間}} \frac{\overbrace{f(x_2)-f(x_1)}^{\text{改變}}}{x_2-x_1}$$

• 形式：一維、二維、三維……。 x：自變數，y：因變數

• 例如：一維，$f(x)=x^2$ 拋物線 $\frac{\Delta f}{\Delta x}=\frac{df}{dx}=\frac{dy}{dx}=f'=y'=\ldots\ldots$

二維，$f(x,y)=x^2y^2+x^2+y^3$，2變數選擇1個，偏微分 $\frac{\Delta f}{\Delta x}=\frac{df(x,y)}{dx}=f_x$

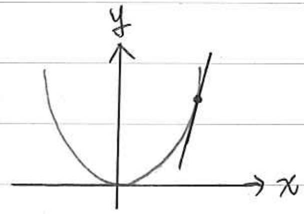
以此類推。

• 微分幾何意義：切線斜率

例如：$f(x)=x^2$ 拋物線

基本定義：$\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}$，瞬間 $x_2\to x_1$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\overset{\text{變形}}{=}\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$



* 方程(式)：原文 equation：字首 equal (=) 等於：含有等於符號數學表達式

• 例如：$ax+b=c$，$ax^2+bx+c=0$，$a_3x^3+a_2x^2+a_1x+a_0=0$，……

※ $x+y+z^2$ 非方程式，為數學表達式

• 重點：求解、驗証解、應用問題。

* 微分方程：含有微分符號的方程式

• 重點：求解、驗証解、應用問題。

• 微分方程為微積分的延伸，微分和積分互逆：解法使用積分技巧。

• 大部分"應用數學"的問題都和"微分方程有關，從真實問題中建立模型(建模)都會連結微分方程，再求解、驗解，解釋回原問題。

• 需要"電腦求解"(數值解)，因此未來有"電腦相關課程：介紹負責數學軟體(maximal)"

↳ e.g. $1\div 3=0.33\ldots$；$1\div 3=\frac{1}{3}$ (符號解)

• 理論重點：解的存在性、唯一性。

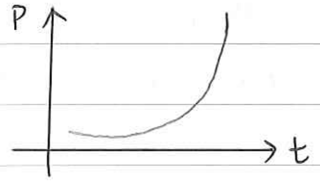
•舉例：應用數學：建立模型(建模)

例子1：人口模型：人口增長和人數成正比

英：馬爾薩斯 1798 提出：$P'(t) = \lambda P(t)$，$\lambda > 0$（微分方程）

求解：$P(t) = P(t_0)e^{\lambda t}$

因為資源有限，所以"解"無法真實描述人口問題（錯誤模型）

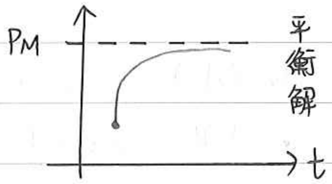


比利時學者：1840 Verhulst 修正此模型，因人口增長不能超過資源上限

logistic 羅吉斯模型 $P'(t) = \lambda P(t)(M - P(t))$（微分方程）

M：資源上限

求解：$P(t) = \dfrac{M}{1+ce^{-\lambda M t}}$



例子2：Lokta - Volterra 狼與兔子模型（1926）

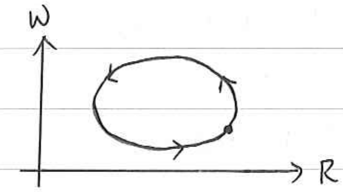
Predator - Prey 獵食者與獵物模型　R：兔子、W：狼

$$\begin{cases} R'(t) = aR(t) - bW(t)R(t) \\ W'(t) = cW(t) - dR(t)W(t) \end{cases}$$

a, b, c, d 為正數（微分方程）

求解：沒有明顯解，用"數值解"取代

例如：將數值解畫在2維圖形上，t：時間



☆期末hw：求狼和兔子生態平衡的係數

• 微積分課本(Calculus)：Ch 6 Differential Equation

§6.1 Slope Field (斜率場) and Euler's Method (歐拉法)

* ① General Solution (一般解) and ② Particular Solution (特別解)

Diff. eq. $y'+2y=0$, general solution is $y=ce^{-2x}$, c is any real number.

• Verify 驗證：代入解到方程，若等號成立，則成立

left hand side LHS = RHS right hand side

$LHS = y'+2y = -2ce^{-2x} + 2\cdot ce^{-2x} = 0 = RHS$, $(y'=-2ce^{-2x})$

e.g 1：Verify which is the solution of $y''-y=0$.

a. $y=\sin x$ b. $y=4e^{-x}$ c. $y=ce^{x}$

a. $LHS = -\sin x - \sin x = -2\sin x \neq 0$ $\therefore LHS \neq RHS$

b. $LHS = 4e^{-x} - 4e^{-x} = 0$ $\therefore LHS = RHS$

c. $LHS = ce^{x} - ce^{x} = 0$ $\therefore LHS = RHS$

• Differential Equation

$\int_0^x f\,dx = F(x) + C$

• General Solution (一般解)：微分積分互逆 —— 不定積分產生常數 C

C 可以是任意數 —— ∵ 許多解 ∴ 有許多曲線

• Initial data (condition 條件) 初始值

帶動固定解 —— C 可以算出

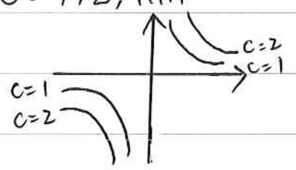
• Particular Solution 特別解：一條曲線

• $y=\frac{c}{x}$ 是 $xy'+y=0$ 的一般解，驗證 $y'=\frac{-c}{x^2}$

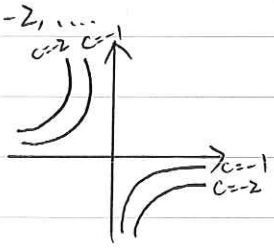
代入 $LHS = x\cdot(\frac{-c}{x^2}) + \frac{c}{x} = 0 = RHS$ $\therefore LHS = RHS$

分析 c 是任意數

c = 1, 2, ...



c = -1, -2, ...



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- 高中物理 — 加速度，自由落體

$S''(t) = -32$ diff. eq.

$S(t) = -16t^2 + C_1 t + C_2$ 一般解，2個常數

$S'(t) = -32t + C_1$ 速度

初速(C_1)，高度(C_2) —— Initial condition 初始條件

例如：$S(0) = 80$，$S'(0) = 64$ 代入一般解得特別解

位置 $S(t) = -16t^2 + 64t + 80$

eg2：Verify $y = cx^3$ is a solution of $xy' - 3y = 0$. Find a particular solution under $y(2) = -3$.

sol： $y' = 3cx^2$， LHS $= x(3cx^2) - 3(cx^3) = 0 =$ RHS ∴ LHS = RHS

$y(2) = c \cdot 8 = -3$， $c = -\frac{3}{8}$， $y = -\frac{3}{8}x^3$

* Slope Field 斜率場

- 微分的幾何意義：切線的斜率

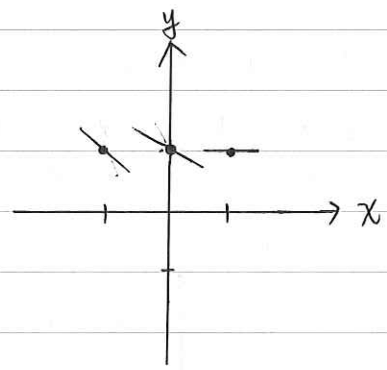
$y' = F(x, y)$ — 每一個點的斜率都可以算出，畫出(一般解)，
畫出的圖就是"斜率場"。

eg3：Sketch (畫) a slope field of $y' = x - y$ for points $(-1,1)$、$(0,1)$ and $(1,1)$.

sol：table (表格)

x	-1	0	1
y	1	1	1
y'	-2	-1	0

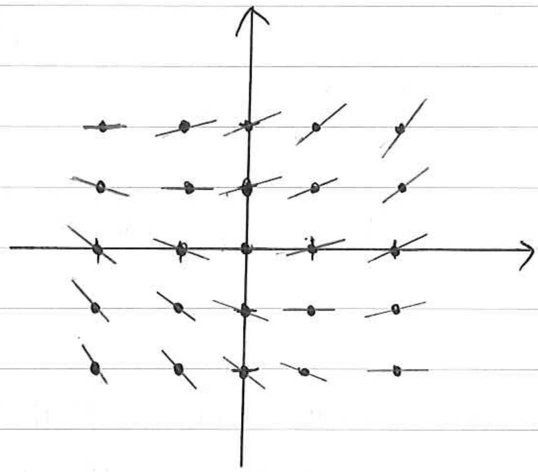
$y' = x - y$



eg4. Sketch the slope field of $y' = x + y$

sol:

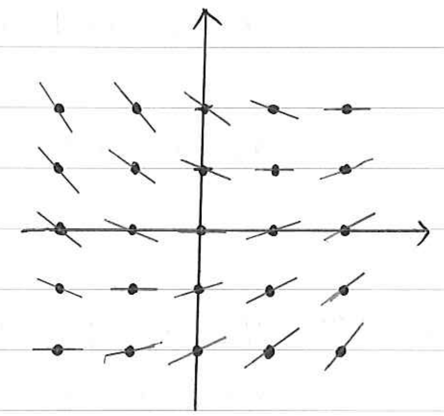
$y'=x+y$	x	-2	-1	0	1	2
y						
2		0	1	2	3	4
1		-1	0	1	2	3
0		-2	-1	0	1	2
-1		-3	-2	-1	0	1
-2		-4	-3	-2	-1	0



eg5. Sketch the slope field of $y' = x - y$

sol:

$y'=x-y$	x	-2	-1	0	1	2
y						
2		-4	-3	-2	-1	0
1		-3	-2	-1	0	1
0		-2	-1	0	1	2
-1		-1	0	1	2	3
-2		0	1	2	3	4



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＊ Euler's Method 歐拉法"數值"(使用數值算)

$y' = F(x, y)$ 初始條件、開始的點(x_0, y_0)，慢慢改變(h)

第一個點 $x_1 = x_0 + h$ $y_1 = y_0 + hF(x_0, y_0)$

第二個點 $x_2 = x_1 + h$ $y_2 = y_1 + hF(x_1, y_1)$

⋮

第n個點 $x_n = x_{n-1} + h$ $y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$ —— 通式

eg 6: Using Euler's Method to approximate (逼近) the particular solution of $y' = x - y$ (斜率) passing (穿過) $(0, 1)$ and $h = 0.1$.
指每移動的點為0.1

sol: $x_1 = 0 + 0.1 = 0.1$ $y_1 = 1 + 0.1(0-1) = 0.9$

$x_2 = 0.1 + 0.1 = 0.2$ $y_2 = 0.9 + 0.1(0.1 - 0.9) = 0.82$

$x_3 = 0.2 + 0.1 = 0.3$ $y_3 = 0.82 + 0.1(0.2 - 0.82) = 0.758 \approx 0.76$

⋮ ⋮

$x_{10} = 0.9 + 0.1 = 1$ $y_{10} = 0.697 \ldots$

Ex. #75 $y' = 3x - 2y$, $y(0) = 3$, $h = 0.05$, Use Euler's Method to get (x_3, y_3).

sol: $x_1 = 0 + 0.05 = 0.05$ $y_1 = 3 + 0.05(0 - 6) = 2.7$

$x_2 = 0.05 + 0.05 = 0.1$ $y_2 = 2.7 + 0.05(3 \times 0.05 - 2 \times 2.7) = 2.4375$

$x_3 = 0.1 + 0.05 = 0.15$ $y_3 = 2.4375 + 0.05(3 \times 0.1 - 2 \times 2.4375)$
$= 2.20875$

A: $(x_3, y_3) = (0.15, 2.20875)$

〈複習〉微分方程 —— 斜率場(普解)

- 給初始條件(initial condition) —— 得到特別解(曲線)
  使用 maxima. 軟體

§6.2 Growth (正比) and Decay (反比)

eg1. Solve $y' = \frac{2x}{y}$.

sol: 直接積分 $\Rightarrow \int y' dx = \int \frac{2x}{y} dx$

先分類(分項), 將相同未知數放一邊

$\Rightarrow yy' = 2x \xRightarrow{\text{变形}} y\frac{dy}{dx} = 2x \Rightarrow y\,dy = 2x\,dx$

$\Rightarrow \int y\,dy = \int 2x\,dx$ 不定積分

$\Rightarrow \frac{1}{2}y^2 = x^2 + C$

$\Rightarrow y^2 - 2x^2 = C$ general solution

- Growth (正比) and Decay (反比) (成長與衰退)

eg2. $\frac{dy}{dt} = ky \begin{cases} k > 0 & \text{growth} \\ k < 0 & \text{decay} \end{cases}$

sol: $\frac{1}{y}dy = k\,dt \Rightarrow \int \frac{1}{y}dy = \int k\,dt \Rightarrow \ln y = kt + C$

$\Rightarrow y = e^{kt+C}$

$\Rightarrow y = e^{kt} \cdot e^{C} = Ce^{kt}$

eg3. Radiactive Decay 輻射衰退

Chernobyle nuclear plant 車諾比核能電廠

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Ex 1 ~ 10

#1 $\frac{dy}{dx} = x+3$

sol:

$dy = (x+3)\,dx$

$\Rightarrow \int dy = \int (x+3)\,dx$

$\Rightarrow y = \frac{x^2}{2} + 3x + C$ ※

#2. $\frac{dy}{dx} = 6-x$

sol:

$dy = (6-x)\,dx$

$\Rightarrow \int dy = \int (6-x)\,dx$

$\Rightarrow y = -\frac{1}{2}x^2 + 6x + C$ ※

#3. $\frac{dy}{dx} = y+3$

sol:

$\frac{dy}{y+3} = dx$

$\Rightarrow \int \frac{1}{y+3}\,dy = \int dx$

$\Rightarrow \ln|y+3| = x + C$

$\Rightarrow y+3 = e^{x} \cdot e^{c} \Rightarrow y = ce^{x} - 3$ ※

#4. $\frac{dy}{dx} = 6-y$

sol:

$\int \frac{1}{6-y}\,dy = \int dx$

$\Rightarrow -\ln|6-y| = x + C$

$\Rightarrow 6-y = e^{-x-c}$

$\Rightarrow y = 6 + e^{-x} \cdot e^{-c} \Rightarrow y = 6 + ce^{-x}$ ※

#5. $y' = \frac{5x}{y}$

sol: $y\frac{dy}{dx} = 5x$

$\Rightarrow y\,dy = 5x\,dx$

$\Rightarrow \int y\,dy = \int 5x\,dx$

$\Rightarrow \frac{1}{2}y^2 = \frac{5}{2}x^2 + C$

$\Rightarrow 5x^2 - y^2 = C$ ※

#6 $y' = \frac{\sqrt{x}}{7y}$

sol:

$7y\,dy = \sqrt{x}\,dx$

$\Rightarrow \int 7y\,dy = \int x^{1/2}\,dx$

$\Rightarrow \frac{7}{2}y^2 = \frac{2}{3}x^{3/2} + C$

$\Rightarrow 4\sqrt{x^3} - 21y^2 = C$ ※

#7. $y' = \sqrt{x}\,y$

sol:

$\frac{1}{y}\,dy = \sqrt{x}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int x^{1/2}\,dx$

$\Rightarrow \ln|y| = \frac{2}{3}\sqrt{x^3} + C$

$\Rightarrow y = Ce^{\frac{2}{3}\sqrt{x^3}}$ ※

#8 $y' = x(1+y)$

sol:

$\int \frac{1}{1+y}\,dy = \int x\,dx$

$\Rightarrow \ln|1+y| = \frac{x^2}{2} + C$

$\Rightarrow y = Ce^{\frac{1}{2}x^2} - 1$ ※

#9 $(1+x^2)y' - 2xy = 0$

sol:

$\frac{1}{y}\,dy = \frac{2x}{1+x^2}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2x}{1+x^2}\,dx$

$\Rightarrow \ln|y| = \ln|1+x^2| + C$

$\Rightarrow y = C(1+x^2)$ ※

#10. $xy + y' = 100x$

sol:

$\frac{dy}{dx} = x(100-y)$

$\Rightarrow \frac{1}{100-y}\,dy = x\,dx$

$\Rightarrow \int \frac{1}{100-y}\,dy = \int x\,dx$

$\Rightarrow -\ln|100-y| = \frac{x^2}{2} + C$

$\Rightarrow y = 100 + Ce^{-\frac{1}{2}x^2}$ ※

分離　　變數

§6.3 Separation of variables and Logistic Equation

※分離變數法 — $M(x) + N(y)\frac{dy}{dx} = 0$

$M(x)$: continuous function of x

$N(y)$: continuous function of y

eg : $x^2 + 3y\frac{dy}{dx} = 0$; $(\sin x)y' = \cos x$; $\frac{xy'}{e^y+1} = 2$

$dy = \frac{-x^2}{3y}dx$; $\cot x\, dx = \frac{\cos x}{\sin x}dx = dy$; $x\frac{dy}{dx} = 2(e^y+1) \Rightarrow dy = \frac{2(e^y+1)}{x}dx$

eg1 : $(x^2+4)\frac{dy}{dx} = xy$

sol : $\frac{1}{y}dy = \frac{x}{x^2+4}dx \Rightarrow \int\frac{1}{y}dy = \int\frac{x}{x^2+4}dx$

integration by substitution 積分代換法

let $u = x^2+4$, $du = 2x\,dx$, $x\,dx = \frac{1}{2}du$

$RHS = \frac{1}{2}\int\frac{1}{u}du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln(x^2+4) + c$

$\Rightarrow \ln|y| = \frac{1}{2}\ln(x^2+4) + c \Rightarrow |y| = e^c\sqrt{x^2+4} = C\sqrt{x^2+4}$

$\Rightarrow y = \pm C\sqrt{x^2+4}$ ※

〈複習〉有系統分類"微分方程"，針對每一類求解。

之前談過 ・斜率場 — 畫出解的曲線

・歐拉法 — 數值法

這兩種方法需要大量勞力(皆可使用電腦軟體協助)

之後求解重點：理論求解

・分離變數法 — 使用微分與積分互逆

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eg 2 : Find the particular solution (特別解) of $xy\,dx + e^{-x^2}(y^2-1)\,dy = 0$ if initial condition (初始條件) $y(0)=1$.

sol: $\frac{y^2-1}{y}\,dy = -e^{x^2}x\,dx \Rightarrow \int y-\frac{1}{y}\,dy = -\frac{1}{2}\int e^{x^2}\,dx^2$

$\Rightarrow \frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^{x^2} + C$

$y(0)=1$ 代入 $\Rightarrow \frac{1}{2} - 0 = -\frac{1}{2}\cdot 1 + C \Rightarrow C=1$

$\therefore y^2 - 2\ln|y| + e^{x^2} = 2 \Rightarrow y^2 - \ln y^2 + e^{x^2} = 2$ ※

eg 3 : Find the equation of the curve that passes (穿過) (1, 3) and has a slope $\frac{y}{x^2}$.

sol: $y' = \frac{y}{x^2}$, $y(1)=3$

分離變數法 : $\int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx \Rightarrow \ln|y| = -x^{-1} + C$

$y(1)=3$ 代入 $\Rightarrow \ln|3| = -1 + C$, $C = \ln|3| + 1$

$\therefore \ln|y| = -\frac{1}{x} + \ln|3| + 1$ ※

* Homogeneous diff. equation 齊次微分方程

- 有時 x, y 不可分離 — 動手腳 (變形) 在 Homogeneous 下可分離變形
- $f(x,y)$ is homogeneous of degree n if $f(tx,ty) = t^n f(x,y)$ (檢驗)

eg 4 : a. $f(x,y) = x^2y - 4x^3 + 3xy^2$ c. $f(x,y) = x + y^2$

b. $f(x,y) = xe^{x/y} + y\sin\frac{x}{y}$ d. $f(x,y) = \frac{x}{y}$

sol: a. $f(tx,ty) = (tx)^2ty - 4(tx)^3 + 3(tx)(ty)^2 = t^3 f(x,y)$ is homogeneous of degree 3

b. $f(tx,ty) = (tx)e^{tx/ty} + ty\sin\frac{tx}{ty} = t f(x,y)$ is homogeneous of degree 1

c. $f(tx,ty) = tx + (ty)^2$ is not homogeneous (無法提出齊次 t)

d. $f(tx,ty) = \frac{tx}{ty} = \underset{t^0 f(x,y)}{\underline{f(x,y)}}$ is homogeneous of degree 0.

* Definition of Homogeneous diff. equation

$M(x,y)\,dx + N(x,y)\,dy = 0$ is homogeneous if M and N are homogeneous of same degree.

eg 5. a. $(x^2+xy)\,dx + y^2\,dy = 0$ yes. degree 2

b. $x^3\,dx = y^3\,dy$ yes. degree 3

c. $(x^2+1)\,dx + y^2\,dy = 0$ not.

(改變變換)

Theorem 6.2 Change of variable for Homongeneous Diff. Equation

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is homongeneous diff. eq.. Let $y = vx$ then the equation can be transformed (變形) into a diff. eq. whose variable are separable. (分離變數法)

eg 6. Find the general solution of $(x^2-y^2)\,dx + 3xy\,dy = 0$

sol: check it is a homogeneous of degree 2

Let $y = vx \Rightarrow dy = x\,dv + v\,dx$ substitution 代入

$(x^2 - v^2x^2)\,dx + 3x\cdot vx\,(x\,dv + v\,dx) = 0$

$\Rightarrow (x^2 - v^2x^2 + 3v^2x^2)\,dx + 3vx^3\,dv = 0$

$\Rightarrow x^2(1+2v^2)\,dx + x^2(3vx)\,dv = 0$

$\Rightarrow \frac{-3v}{1+2v^2}\,dv = \frac{1}{x}\,dx \quad \Rightarrow -3\int \frac{v}{1+2v^2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \frac{-3}{4}\int \frac{1}{1+2v^2}\,d(1+2v^2) = \int \frac{1}{x}\,dx$

$\Rightarrow -\frac{3}{4}\ln(1+2v^2) = \ln|x| + c$

$\Rightarrow -3\ln(1+2v^2) = 4\ln|x| + c$

$\Rightarrow e^{\ln(1+2v^2)^{-3}} = e^{\ln x^4 + c}$

$\Rightarrow (1+2v^2)^{-3} = x^4 e^c$

$\Rightarrow x^4 = c(1+2v^2)^{-3}$

$(y = vx,\ v = \frac{y}{x})$ $\Rightarrow x^4 = c\left(1+2\left(\frac{y}{x}\right)^2\right)^{-3}$

$\Rightarrow \left(1+\frac{2y^2}{x^2}\right)^3 x^4 = c \quad \Rightarrow (x^2+2y^2)^3 = cx^2$ ※

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• 整理 1、確認齊次微分方程
2、令 $y = vx$, $dy = x\,dv + v\,dx$ 代入
3、分離變數法求解
4、$v = \frac{y}{x}$ 代回解整理

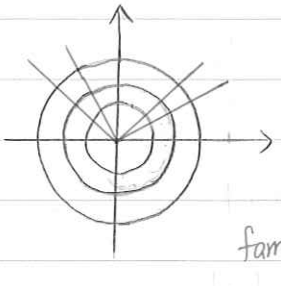
Ex #39 $y' = \frac{x+y}{2x}$

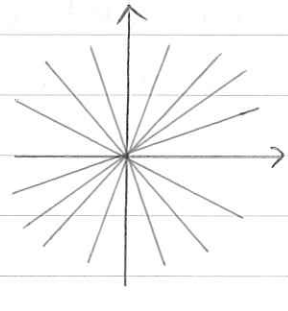
sol: $y = vx$, $dy = x\,dv + v\,dx$

$$x\,dv + v\,dx = \left(\frac{x+vx}{2x}\right)dx \Rightarrow x\,dv + v\,dx = \frac{1+v}{2}\,dx$$
$$\Rightarrow x\,dv = \frac{1-v}{2}\,dx$$
$$\Rightarrow \frac{1}{2x}\,dx = \frac{1}{1-v}\,dv$$
$$\Rightarrow \frac{1}{2}\int \frac{1}{x}\,dx = -\int \frac{1}{1-v}\,d(1-v)$$
$$\Rightarrow \frac{1}{2}\ln|x| = -\ln|1-v| + c$$
$$\Rightarrow \frac{1}{2}\ln|x| = -\ln\left|1-\frac{y}{x}\right| + c \quad \#$$

〈複習〉分類、分辨微分方程，針對每一類求解，驗證解
目前已介紹：斜率場、分離變數法、齊次微分方程

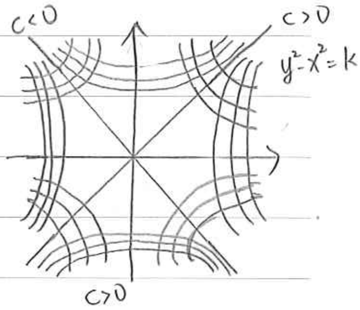
* 介紹曲線族 family of curves.
eg $x^2+y^2 = c$, c: 任意常數 eg. $y = kx$, k = 任意常數


family of curves


family of lines

The two families are mutually orthognal (兩兩垂直)
⇓
斜率相乘 = -1

eg 8. Describe the orthognal trajectories for the family of curves given by $y=\frac{c}{x}$ for $c \neq 0$.



sol: 了解 $y=\frac{c}{x}$ 兩兩垂直的圖線色的線

先找 $y=\frac{c}{x}$ 的斜率 (微分=斜率)

$\Rightarrow xy=c \Rightarrow y\,dx+x\,dy=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x}$ (斜率)

$\because m_1 \cdot m_2=-1$ (互相垂直的斜率)

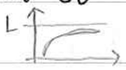
$\therefore$ 垂直曲線族的斜率為 $\frac{dy}{dx}=\frac{x}{y}$，解出微分方程

$\Rightarrow y\,dy=x\,dx \Rightarrow \int y\,dy=\int x\,dx \Rightarrow \frac{1}{2}y^2=\frac{1}{2}x^2+c \Rightarrow y^2-x^2=k$

特別的是 $k=0 \Rightarrow y^2-x^2=0$

eg 9. Solve the logistic differential equation $\frac{dy}{dt}=ky(1-\frac{y}{L})$ 羅吉斯人口模型

L: 封閉環境可容納最大資源



sol: $\frac{1}{y(1-\frac{y}{L})}dy=k\,dt \Rightarrow \int \frac{1}{y(1-\frac{y}{L})}dy=\int k\,dt$

$\Rightarrow \int \frac{L}{y(L-y)}dy=\int k\,dt$

$\frac{L}{y(L-y)}=\frac{A}{y}+\frac{B}{L-y}$

$A=1,\ B=1$

$\Rightarrow \int (\frac{1}{y}+\frac{1}{L-y})dy=\int k\,dt$

$\Rightarrow \ln|y|-\ln|L-y|=kt+c$

$\Rightarrow \ln\left|\frac{y}{L-y}\right|=kt+c$

$\Rightarrow \ln\left|\frac{L-y}{y}\right|=-kt+c \Rightarrow \frac{L-y}{y}=ce^{-kt} \Rightarrow y=\frac{L}{1+ce^{-kt}}$ ※

L為資源上限 $L>y \Rightarrow \frac{y}{L-y}=e^{kt}\cdot c$

$\Rightarrow y=(L-y)ce^{kt}$

$\Rightarrow y(1+ce^{kt})=Lce^{kt}$

$\Rightarrow y=\frac{Lce^{kt}}{1+ce^{kt}}$ ※

§6.4 First order Linear Diff. Equation. 一階線性微分方程

$\frac{dy}{dx} + p(x)\,y = Q(x)$

- standard form 標準式：$y' + p(x)y = Q(x)$ ，P and Q are continuous function of x.

將此乘上積分因子

- idea 定義：$u(x) = e^{\int p(x)dx}$ integrating factor（積分因子）

$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)\,y = Q(x)e^{\int p(x)dx}$

$\Rightarrow [ye^{\int p(x)dx}]' = Q(x)e^{\int p(x)dx} \Rightarrow \int ye^{\int p(x)dx}\,dy = \int Q(x)e^{\int p(x)dx}\,dx$

$\Rightarrow y = \frac{1}{e^{\int p(x)dx}}\int Q(x)e^{\int p(x)dx}\,dx$

$\Rightarrow y = \frac{1}{u(x)}\int Q(x)\,u(x)\,dx$ （公式）

eg1. Solve $y' + y = e^x$

sol：$P(x) = 1$ ，$Q(x) = e^x$ ，$u(x) = e^{\int dx} = e^x$ （By 標準式．積分因子）

公式：$y = \frac{1}{e^x}\int e^x\cdot e^x\,dx = \frac{1}{e^x}\int e^{2x}dx = \frac{1}{e^x}(\frac{1}{2}e^{2x} + c) = \frac{1}{2}e^x + ce^{-x}$ ※

推理：$(\frac{dy}{dx} + y = e^x) \overset{\times e^x}{\Rightarrow} [e^x\frac{dy}{dx} + ye^x] = e^{2x} \Rightarrow (ye^x)' = e^{2x}$

$\Rightarrow \int [ye^x]'\,dy = \int e^{2x}dx \Rightarrow ye^x = \frac{1}{2}e^{2x} + c$

$\Rightarrow y = \frac{1}{2}e^x + ce^{-x}$

Theorem 6.3 $y' + p(x)y = Q(x)$. Let $u(x) = e^{\int p(x)dx}$. Then solution

$ye^{\int p(x)dx} = \int Q(x)e^{\int p(x)dx}\,dx + c$

eg 2. Solve $xy' - 2y = x^2$

sol：分辨：非標準式

$y' - \frac{2}{x}y = x$ ，$p(x) = -\frac{2}{x}$ ，$Q(x) = x$ ，$u(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$

公式 $\Rightarrow y = x^2\int x\cdot\frac{1}{x^2}\,dx \Rightarrow y = x^2\int\frac{1}{x}\,dx$

$\Rightarrow y = x^2\ln|x| + c$ ※

eg3. Solve $y' - y\tan t = 1$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

sol: $P(x) = -\tan t$, $Q(x) = 1$, $u(t) = e^{\int P(t)dt} = e^{-\int \tan t\, dt} = e^{\ln|\cos t|} = |\cos t|$

$(\int \frac{\sin t}{\cos t} dt = -\int \frac{1}{\cos t} d(\cos t) = -\ln|\cos t|)$.

同乘 $u(t) \Rightarrow \cos t\, y' - \cos t\, y \tan t = \cos t$

$\Rightarrow [y\cos t]' = \cos t$

積分 $\Rightarrow y\cos t = \int \cos t\, dt = \sin t + c$

$\Rightarrow y = \frac{\sin t}{\cos t} + \frac{c}{\cos t} = \tan t + c\sec t.$

* Bernoulli Equation 白努利方程

- 標準式 : $y' + P(x)y = Q(x)y^n$

$n=0$ $y' + P(x)y = Q(x)$ 一階線性

$n=1$ $y' + P(x)y = Q(x)y \Rightarrow y' = \frac{dy}{dx} = (Q(x) - P(x))y$

$\Rightarrow \frac{1}{y}dy = (Q(x) - P(x))dx$ 可分離

$n \geq 2$ $y^{-n}y' + p(x)^{1-n} = Q(x) \Rightarrow (1-n)[y^{-n}y' + P(x)y^{1-n}] = (1-n)\cdot Q(x)$

Let $z = y^{1-n}$, $\Rightarrow z' + (1-n)p(x)z = (1-n)Q(x)$

$z' = (1-n)y^{-n}\cdot y'$ 對 z 來說 : 一階線性微分方程

- 公式 : $y^{1-n} e^{\int(1-n)p(x)dx} = \int (1-n)Q(x)e^{\int(1-n)P(x)dx} dx + c$

eg7. Solve $y' + xy = xe^{-x^2}y^{-3}$

sol: 分辨 : 白努利方程

Let $z = y^{1-(-3)} = y^4$, $z' = 4y^3y'$

$y' + xy = xe^{-x^2}y^{-3} \Rightarrow 4y^3y' + 4y^4x = 4xe^{-x^2} \Rightarrow z' + 4zx = 4xe^{-x^2}$ (z的一階線性微分方程)

using $p(x) = 4x \Rightarrow \int p(x)dx = \int 4x\, dx = 2x^2$

$z' + 4xz = 4xe^{-2} \xRightarrow{\times e^{\int p(x)dx}} z'e^{2x^2} + 4xze^{2x^2} = 4xe^{x^2} \Rightarrow \frac{d}{dx}[ze^{2x^2}] = 4xe^{x^2}$

同積分 $\Rightarrow ze^{2x^2} = \int 4xe^{x^2} \Rightarrow ze^{2x^2} = 2e^{x^2} + c \Rightarrow z = 2e^{-x^2} + ce^{-2x^2}$

substituting $z = y^4 \Rightarrow y^4 = 2e^{-x^2} + ce^{-2x^2}$ (一般解)

影片：Bernoulli 方程式解法

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$$y' + P(x)y = g(x)y^a \,,\ a \neq 0, 1 \quad \text{Bernoulli Equation}$$

$\because a = 0 \Rightarrow y' + p(x)y = g(x)$

$a = 1 \Rightarrow y' + p(x)y = g(x)y \Rightarrow y' + [P(x) - g(x)]\,y = 0$

$\rangle\ y' + p(x)y = g(x)$

$$y' + p(x)\,y = g(x)\,y^a$$

$\div y^a \Rightarrow \underbrace{\frac{1}{y^a}y'}_{y^{-a}y' \,=\, \frac{1}{1-a}\frac{du}{dx}} + p(x)\frac{y}{y^a} = g(x)$ ，令 $\frac{y}{y^a} = u(x)$

$\because u = \frac{y}{y^a} = y^{1-a}$

$\therefore \frac{du}{dx} = \frac{d(y^{1-a})}{dx} = \frac{d(y^{1-a})}{dx}\frac{dy}{dx} = (1-a)\underbrace{y^{1-a-1}\frac{dy}{dx}}_{y^{-a}y'}$

$\Rightarrow y^{-a}y' = \frac{1}{1-a}\frac{du}{dx}$

$$\frac{1}{y^a}y' + p(x)\frac{y}{y^a} = g(x)$$

$$\Rightarrow \frac{1}{1-a}\frac{du}{dx} + p(x)\,u = g(x)$$

$$\Rightarrow \boxed{\frac{du}{dx} + (1-a)p(x)\,u = (1-a)\,g(x)}$$

〈複習〉解微分方程方法

- 斜率場 — 解曲線
- 紙筆解法：1、分離變數法 2、齊次微分方程
  3、一階線性微分方程 4、白努利微分方程
  5、正合(exact)微分方程

- 宏觀數學知識 1、概念性知識 ⟶ 思考
  2、程序性知識 ⟶ 計算

- Bernoulli Equation 白努利 $y'+P(x)y=Q(x)y^n$

$n=0$ 一階線性 ； $n=1$ 分離變數法

$n>1$ 白努利代換找出 n，$1-n$，令 $z=y^{1-n}$ 微分代換

eg. $xy'+6y=3xy^{\frac{4}{3}}$

sol：標準式 $y'+\frac{6}{x}y=3y^{4/3}$ —— 白努利

分析 $n=\frac{4}{3}$，$1-\frac{4}{3}=-\frac{1}{3}$，令 $z=y^{-1/3}$，$z'=-\frac{1}{3}y^{-4/3}y'$，$y=z^{-3}\Rightarrow$

$y'=-3z^{-4}z'$，標準式 $\times(-\frac{1}{3})\Rightarrow -\frac{1}{3}y'-\frac{2}{x}y=-y^{4/3}$

$\times(y^{-4/3})\Rightarrow -\frac{1}{3}y^{-4/3}y'-\frac{2}{x}y^{-1/3}=-1$

代換成 $z\Rightarrow z'-\frac{2}{x}z=-1$ 一階線性

積分因子 $u(x)=e^{\int-\frac{2}{x}dx}=x^{-2}$

$z=\frac{1}{u(x)}\int Q(x)u(x)\,dx=x^2\int -x^{-2}\,dx=-x^2(-x^{-1}+c)=x+cx^2$

$z=x+cx^2$ 代回 $y\Rightarrow y^{-1/3}=x+cx^2\Rightarrow y(x)=\frac{1}{(x+cx^2)^3}$

Ex 51. $xy'+y=xy^3$

sol: $y'+\frac{1}{x}y=y^3$，let $z=y^{1-3}=y^{-2}$，$z'=-2y^{-3}y'$

$-2y^{-3}y'-\frac{2}{x}y^{-2}=-2\Rightarrow z'-\frac{2}{x}y^{-2}=-2$

$u(x)=e^{\int-\frac{2}{x}dx}=\frac{1}{e^{\ln x^2}}=x^{-2}$

$z=x^2\int -2x^{-2}\,dx=x^2(2x^{-1}+c)$

$\Rightarrow y^{-2}=x^2(\frac{2}{x}+c)$

$\Rightarrow y=\frac{1}{x\sqrt{\frac{2}{x}+c}}$ ✫

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* Exact Diff. Equation 正合微分方程

- 隱函數微分：有些函數 $y(x)$ 藏在方程式中

eg. $x^2+y^2+x^2y^3=4$ ，y 函數解不出

$$\frac{d}{dx} \Rightarrow 2x+2y\cdot\frac{dy}{dx}+2xy^3+x^2\cdot 3y^2\frac{dy}{dx}=0$$

$$\Rightarrow (2y+3x^2y^2)\frac{dy}{dx}=-2x-2xy^3$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2x-2xy^3}{2y+3x^2y^2} \quad ※$$

類似推廣至微分方程 $F(x, y(x))=C$

i.e. $M(x,y)+N(x,y)\frac{dy}{dx}=0$ where $M(x,y)=F_x$, $N(x,y)=F_y$.

有時候 $M(x,y)\,dx+N(x,y)\,dy=0$ exact 正合

剛剛好是某個 $F(x,y(x))=C$ 方程式的隱函數微分

- Exact diff. equation 正合 (恰當. 剛剛好)

剛剛好是某個方程式的隱函數，微分形成的微分方程

- Differential form : $M(x,y)\,dx+N(x,y)\,dy=0$

If $\exists$ $f(x,y)$ such that $\frac{\partial F}{\partial x}=M$ and $\frac{\partial F}{\partial y}=N$, then $F(x,y)=C$ implicitly (隱) defines a general solution.

- $M(x,y)\,dx+N(x,y)\,dy=0$ is an exact diff. eq. .

The differential $dF=F_x\,dx+F_y\,dy$ of $F(x,y)$ is "exactly" $M\,dx+N\,dy$.

- 隨便搭一組 $M\,dx+N\,dy$ 未必是"正合"，需要判別式 $M_y=F_{xy}=F_{yx}=N_x$.

i.e. 不是每個微分方程皆是正合 ∴需要一個判別式

If $M_y=F_{xy}=F_{yx}=N_x$, then $M\,dx+N\,dy=0$ is exact.

eg 8. $y^3\,dx + 3xy^2\,dy = 0$ exact ?

sol: 驗證 $M_y = N_x$, $M_y = 3y^2$, $N_x = 3y^2$ $\Rightarrow M_y = N_x$ $\therefore$ exact.

$F(x,y) = xy^3$ is a general solution.

$F_x = y^3 = M$, $F_y = 3xy^2 = N$

另一個觀察 $y^{-2}(y^3\,dx + 3xy^2\,dy) = 0$

$\Rightarrow y\,dx + 3x\,dy = 0$ — exact ?

驗證 : $M_y = 1$, $N_x = 3$ $\therefore$ isn't exact.

• 兩個等價的微分方程 $\begin{cases} y^3\,dx + 3xy^2\,dy = 0 \\ y\,dx + 3x\,dy = 0 \end{cases}$ 有共同解 $xy^3 = c$,

但一個是正合，另一個不是。

Theorem 1 正合微分方程

$M(x,y)$ and $N(x,y)$ are continuous and have continuous first order partial derivatives (偏微分) in the open rectangle $R: a < x < b$, $c < y < d$.

Then, $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact $\Leftrightarrow M_y = N_x$.

That is, $\exists$ a function $F(x,y)$ defined on R with $F_x = M$ and $F_y = N$.

eg 9: Solve $(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$

sol. check exactness. $M_y = 6x - 3y^2$, $N_x = 6x - 3y^2$. $M_y = N_x$ $\therefore$ exact.

$\frac{\partial F(x,y)}{\partial x} = M = 6xy - y^3 \Rightarrow F(x,y) = \int 6xy - y^3\,dx = 3x^2y - xy^3 + g(y)$

$\frac{\partial F(x,y)}{\partial y} = N = 3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$

$\Rightarrow g'(y) = 4y \Rightarrow g(y) = 2y^2 + C$

solution is $F(x,y) = 3x^2y - xy^3 + 2y^2 = C$ ※

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Ex : Solve $(3x^2y+3xy^2)\,dx+(x^3+3x^2y)\,dy=0$

sol: 正合? $M_y=3x^2+6xy$, $N_x=3x^2+6xy$, $M_y=N_x$ $\therefore$ exact.

$\frac{\partial F(x,y)}{\partial x}=M=3x^2y+3xy^2 \Rightarrow F(x,y)=\int 3x^2y+3xy^2\,dx=x^3y+\frac{3}{2}x^2y^2+g(y)$

$\frac{\partial F(x,y)}{\partial y}=N=x^3+3x^2y+g'(y)=x^3+3x^2y$

$\Rightarrow g'(y)=0$, $g(y)=c$

$\therefore F(x,y)=x^3y+\frac{3}{2}x^2y^2=c$

〈註〉解微分方程,常數C經常變,所以只有單一變數時,$c=-c=e^c=\ln|c|$
〈註〉一階微分方程,搭配一個常數C
二階微分方程,搭配二個常數 C_1. C_2.

*退化二階微分方程 Reducible 2nd-Order equation.

A 2nd-order diff. eq involves y'', y', y, x

$F(x,y,y',y'')=0$ x:自變數, $y.y'.y''$ 因變數

If x or y is missing, it is reduced (退化) by a substition (代換) to a 1st order diff. eq.

- Case 1. y missing $F(x,y',y'')=0$

代換令 $p=y'=\frac{dy}{dx}$, $p'=\frac{dp}{dx}=y''$

$\Rightarrow F(x,p,p')=0$ 一階微分方程

$\Rightarrow y=\int y'\,dx=\int p(x,C_1)\,dx+C_2$

eg 10. Solve $xy''+2y'=6x$ 無y的退化二階方程

sol: check 2階微方但沒有y $\Rightarrow F(x,y',y'')$

令 $p=y'$, $p'=y''$ 代換 $\Rightarrow xp'+2p=6x$ 一階線性微方 $\xRightarrow{\text{標準化}} p'+\frac{2}{x}p=6$

積分因子 $u(x)=e^{\int p(x)dx}=e^{\int \frac{2}{x}dx}=x^2$

$\Rightarrow p=\frac{1}{x^2}\int 6x^2\,dx=2x+\frac{C_1}{x^2}=y'$

$\Rightarrow y(x)=x^2+\frac{C_1}{x}+C_2$

Ex : Solve $xy'' + y' = 4x^3$ $\quad F(x, y', y'')$

sol: 令 $p = y'$, $p' = y''$ 代換 $\Rightarrow xp' + p = 4x^3 \xRightarrow{\text{標準化}} p' + \frac{1}{x}p = 4x^2$

積分因子 $u(x) = e^{\int \frac{1}{x} dx} = x$

$p = \frac{1}{x}\int 4x^2 \cdot x \, dx = \frac{1}{x}(x^4 + C_1) = y'$

$y(x) = \int x^3 + \frac{C_1}{x} \, dx = \frac{1}{4}x^4 + C_1 \ln|x| + C_2$ ※

〈複習〉解一階微分方程

- 退化二階微分方程 (階至二個一階) $\quad F(x, y, y', y'') = 0$

Case 1 : y missing

Case 2 : x missing , $F(y, y', y'') = 0$ 代換法

令 $p = y'$, $y'' = p' = \frac{dp}{dx} = \frac{dp}{dy} \times \frac{dy}{dx} = p\frac{dp}{dy} \Rightarrow F(y, p, p\frac{dp}{dy}) = 0$

If solving for a general solution $p(y, C_1)$,

$x(y) = \int \frac{dx}{dy} dy = \int \frac{1}{dy/dx} dy = \int \frac{1}{p} dy = \int \frac{1}{p(y, C_1)} dy + C_2$

If $P = \int \frac{1}{p} dy$ can be evaluated , then $\exists$ an implicit solution , $x(y) = P(y, C_1) + C_2$

程序性知識

eg 11. Solve $yy'' = (y')^2$ $\quad$ (x-missing)

sol: 令 $p = y'$, $y'' = p \cdot \frac{dp}{dy}$ 代換 $\Rightarrow yp\frac{dp}{dy} = p^2 \Rightarrow \frac{1}{p}dp = \frac{1}{y}dy$

$\Rightarrow \int \frac{1}{p} dp = \int \frac{1}{y} dy \Rightarrow \ln p = \ln y + C_1$ ($\because y > 0$, $p = y' > 0$)

$\Rightarrow p = ye^{C_1} = C_1 y$

Hence , $\frac{dx}{dy} = \frac{1}{p} = \frac{1}{C_1 y} \Rightarrow \int C_1 dx = \int \frac{1}{y} dy \Rightarrow C_1 x = \ln y + C_2$

$\Rightarrow e^{C_1 x} = e^{\ln y} \cdot e^{C_2}$

$\Rightarrow y(x) = e^{C_1 x - C_2} = Ae^{Bx}$

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